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An algebraic and geometric approach to non-bijective quadratic transformations

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Abstract. This work is concerned with an extension of the so-called Levi-Civita and Kustaanheimo-Stiefel transformations. The extension is achieved along two lines. Firstly, we examine the latter transformations in other dimensions than the ones originally considered by, on the one hand, Levi-Civita and, on the other hand, by Kustaanheimo and Stiefel. Secondly, we pass from the compact to the non-compact case. This leads to quadratic non-bijective transformations that we refer to as Hurwitz (or Kustaanheimo-Stiefel-like) and quasiHurwitz (or Levi-Civita-like) transformations. The Hurwitz and quasiHurwitz transformations are introduced and studied in an algebraic framework which relies on the use of (eight-dimensional) Cayley-Dickson algebras. An explicit formulation of the Hurwitz and quasiHurwitz transformations is also given in terms of Clifford algebras. The Hurwitz transformations are investigated from a geometrical viewpoint. Indeed, they are connected to Hopf and 'pseudoHopf' fibrations. Finally, some differential aspects of the Hurwitz and (to a lesser extent) quasiHurwitz transformations are developed in view of future physical applications.

1. Introduction

Quadratic non-bijective transformations turn out to be of paramount importance in mathematical and theoretical physics and in quantum chemistry. As a first example, everyone is familiar with the (two-dimensional) conformal transformation (an $\mathbb{R}^2 \to \mathbb{R}^2$ surjection with discrete kernel), also referred to as the Levi-Civita (LC) transformation. By way of illustration, this well known transformation has been used for the restricted three-body problem [1] and for problems relative to $\mathbb{R}^2$ and $\mathbb{R}^3$ hydrogen atoms [2-5]. To be more precise, the LC transformation makes it possible to convert, in Schrödinger equation [2, 5] and in Feynman path integral [4] formulations, the problem of an $\mathbb{R}^2$ hydrogen atom into the one of an $\mathbb{R}^2$ isotropic harmonic oscillator subjected to a constraint; in addition, the LC transformation has been used in a classical approach to the quadratic Zeeman Hamiltonian for the $\mathbb{R}^3$ hydrogen atom [3].

As a second example, we mention a transformation which is often referred to as the Kustaanheimo-Stiefel (KS) transformation (an $\mathbb{R}^4 \to \mathbb{R}^3$ surjection with continuous kernel) and which naturally arises in the theory of spinors (see for instance [6]). Such a transformation has been the object in recent years of considerable interest in celestial mechanics, theoretical physics and theoretical chemistry. It has been used by Kustaanheimo and Stiefel [6, 7] for regularisation at the origin of the three-dimensional Kepler problem. This transformation has also been considered independently by Ikeda and Miyachi† [8] in the framework of a unified treatment of the $\mathbb{R}^4$ isotropic harmonic oscillator, the $\mathbb{R}^3$ hydrogen atom and the $\mathbb{R}^3$ spherical rotator. Further works on the connection between the $\mathbb{R}^3$ hydrogen atom with Coulomb potential only and the $\mathbb{R}^4$ isotropic harmonic oscillator (or, equivalently, a coupled pair of $\mathbb{R}^2$ isotropic harmonic oscillators) have been conducted using the KS transformation in a (Schrödinger) partial differential equation approach [9-17], a (Feynman) path integral approach [18-23] and a (Weyl-Wigner-Moyal) phase space approach [24]. It has also been shown that the problem of an $\mathbb{R}^3$ hydrogenlike atom in an electric or a strong magnetic field can be converted, by means of the KS transformation, into the problem of an $\mathbb{R}^4$ isotropic anharmonic oscillator presenting anharmonicity of degree 4 and 6 respectively and subjected to a constraint [25]. The KS transformation has also been used to tackle the problem of an $\mathbb{R}^3$ hydrogen atom in the presence of a time-dependent electric field [26]. Furthermore, the interest of the KS transformation has been pointed out for potentials of relevance in various problems from quarks to atoms and molecules as, for example, the Hartmann potential (of relevance for ring-shaped molecules) and the Kratzer potential (of relevance for vibration-rotation spectroscopy of molecules) [27]. To close this (probably not exhaustive) review, we mention that the KS transformation has been used in connection with a quantum theory of infinite component fields [10], with a complete geometrical description of the magnetic monopole of Wu and Yang [28], with a characterisation of a new class of instantons [29] and with a study of supersymmetries for particles in a Coulomb potential [30].

The KS transformation, which is not an extension of the LC transformation (as noted in [27]), and the LC transformation both have counterparts in $n=2$, 4 and 8 dimensions. Many of these transformations are closely related to the classical Hopf bundles [31-33]. (As is well known, the KS transformation corresponds to the $S^3/S^1 = S^2$ Hopf fibration.) In this respect, Polubarinov has studied quadratic transformations corresponding primarily to the Hopf fibrations on spheres [32]. His study is developed in terms of spinors and hypercomplex numbers with the aim of obtaining Fierz identities for transforming classical or quantum Lagrangians and Hamiltonians. Quite recently, a tentative classification of quadratic transformations not necessarily related to Hopf maps has been proposed [33].

It is one of the aims of this paper to investigate, in a general algebraic framework, usual (i.e. compact) as well as generalised (i.e. compact and non-compact) quadratic transformations. This framework relies on the use of real Cayley-Dickson algebras [34, 35] in eight dimensions. The latter algebras turn out to be particularly relevant for physical applications (see [35]). They cover the classical and hyperbolic octonions which have been used by Moffat [36] for describing a non-symmetric theory of gravitation in four dimensions. The four-dimensional subalgebras of the eight-dimensional Cayley-Dickson algebras are the algebras consisting either of classical quaternions or of hyperbolic quaternions. Classical and hyperbolic quaternions have been used by Jantzen [37] to study spacetime symmetries in cosmological problems. The two-dimensional subalgebras of the four-dimensional Cayley-Dickson algebras include the algebra of complex numbers and the one of hyperbolic complex numbers. The hyperbolic complex numbers have been used to describe pseudoconformal transformations in two-dimensional hydrodynamics [38]. All the aforementioned algebras in

† These authors employ a parametrisation of $\mathbb{R}^4$ from which the Kustaanheimo-Stiefel transformation arises quite naturally.

$n=2$, 4 and 8 dimensions are *composition algebras* and either *division algebras* or *singular* (following the terminology of Ilamed and Salingaros [39]) *division algebras.* A general study and construction of the Cayley-Dickson algebras in any dimension is given in the papers by Wene [35] who exhibits a close connection between the latter algebras and the Clifford algebras.

We start in § 2 from the eight-dimensional Cayley-Dickson algebras and define anti-involutions of these algebras and of their subalgebras. This leads to a decomposition of the Cayley-Dickson algebras of dimensions $n=2$, 4 and 8 in the form $H \oplus M$, where H is a Lie-admissible subspace and M an (ordinary) subspace (which may not span a subalgebra). These anti-involutions allow us to define in § 3 what we call Hurwitz transformations. These transformations correspond to maps from the considered Cayley-Dickson algebras onto their subspaces M. A first property of the Hurwitz transformations concerns a quadratic relation which generalises a relation occurring in the celebrated Hurwitz factorisation problem [40] on the sum of $n=2$, 4 or 8 squared numbers. A second property deals with their invariance under the Lie group G constructed from the set of generators of H and even with their invariance under $G/\mathbb{Z}_2$. Then, we identify the various Hurwitz transformations to quadratic non-bijective transformations with kernel $G/\mathbb{Z}_2$. An explicit formulation of the Hurwitz transformations in terms of Clifford algebras is also given in § 3.

In the four-dimensional compact case, the so-called Hurwitz transformations are nothing but alternative presentations of the KS transformation. In contradistinction, in the two-dimensional compact case, it is not possible to recover the LC transformation in the framework of the Hurwitz transformations. In this sense, the KS transformation is not an extension of the LC transformation as noted above. In § 4 we introduce what we call quasiHurwitz transformations. These transformations constitute compact and non-compact generalisations in $n=2$, 4 and 8 dimensions of the LC transformation. (The quasiHurwitz transformations may be defined equally well in arbitrary $2m$ dimensions.) They are built as quadratic transformations on two-, four- and eight-dimensional Cayley-Dickson algebras (a fact which justifies the name quasiHurwitz) with a discrete kernel $\mathbb{Z}_2$. (A particular example of these quasiHurwitz transformations has been already worked out by Kibler and Négadi in the four-dimensional compact case [27].) A Clifford formulation of the quasiHurwitz transformations is also given in § 4.

A second goal of this work is to investigate the Hurwitz transformations in a geometric framework. This is achieved in § 5 where the invariance under the group $G/\mathbb{Z}_2$ of the Hurwitz transformations leads to fibre bundles. More precisely, we are led (i) to classical Hopf bundles (i.e. with fibrations on spheres) when choosing the algebras of complex numbers, quaternions and octonions (a result that is known) and (ii) to pseudoHopf bundles (i.e. with fibrations on hyperboloids) when choosing the algebras of hyperbolic complex numbers, hyperbolic quaternions and hyperbolic octonions (a result which is apparently new). The pseudoHopf bundles admit compact and non-compact fibres and have as base spaces either single-sheeted hyperboloids or two-sheeted hyperboloids.

The Hurwitz transformations (which involve the KS transformation) and the quasi-Hurwitz transformations (which involve the LC transformation) can be used to relate (elliptic or hyperbolic) differential operators in different dimensions. This point is examined in § 6 where we relate several hyperbolic differential operators in three and four dimensions as well as in five and eight dimensions. This gives the key for switching to physical applications. In particular, we deal in § 7 with applications to non-compact sigma models on curved spaces (cf [41] for non-linear sigma models on spheres). Generally speaking, each application is concerned with a dimensional reduction process (cf the dimensional reduction of Hamiltonian systems in [42]). This may be transcribed, for quantum mechanical systems, in terms of invariance and non-invariance algebras as shown in § 7. A typical example is provided with the $\mathbb{R}^3$ ($\mathbb{R}^5$) hydrogen atom whose non-invariance algebra so(4, 2) (so(6, 2)) may be obtained from a dimensional reduction approach to the $\mathbb{R}^4$ ($\mathbb{R}^8$) isotropic harmonic oscillator.

This article contains two appendices. In appendix 1, we give an explicit realisation of the Clifford matrices Γ_k of orders 8, 4 and 2 occurring in § 3 and briefly discuss the Dirac groups associated to the Clifford algebras of degree $2m-1$. In appendix 2, we construct the 'inverses' of the Hurwitz transformations in $n=2$, 4 and 8 dimensions.

2. Eight-dimensional Cayley–Dickson algebras

2.1. Description of the algebras and of their subalgebras

The eight-dimensional Cayley-Dickson algebras are generated by a set of seven generators $e_1, e_2, \ldots, e_7$ which obey the multiplication law defined in table 1, where each c_i ($i=1,2,3$) stands for 1 or -1. In other words, we have the rule

$$e_k e_l = -g_{kl} + \sum_{m=1}^{7} a_{kl}{}^m e_m \qquad (k \text{ and } l = 1, 2, \ldots, 7) \tag{1}$$

where the g_{kl} are the matrix elements of

$$g = \operatorname{diag}(-c_1, -c_2, c_1c_2, -c_3, c_1c_3, c_2c_3, -c_1c_2c_3) \tag{2a}$$

and the constants $a_{kl}{}^m = -a_{lk}{}^m$ directly follow from table 1. Let us remark that the tensor defined by

$$a_{klm} = \sum_{n=1}^{7} g_{mn} a_{kl}{}^n \tag{2b}$$

is totally antisymmetric. We shall denote by $A(c_1, c_2, c_3)$ the eight-dimensional Cayley-Dickson algebra on $\mathbb{R}$ responding to the (standard) multiplication law described by table 1. The unit element e_0 of $A(c_1, c_2, c_3)$ is simply denoted 1 as is implicit in (1). A general element u in $A(c_1, c_2, c_3)$ is

$$u = u_0 + \sum_{k=1}^{7} u_k e_k \qquad u_\alpha \in \mathbb{R} \qquad (\alpha = 0, 1, \ldots, 7). \tag{3}$$

Table 1. Table of the products $e_k e_l$ for the Cayley-Dickson algebra $A(c_1, c_2, c_3)$. For instance $e_2 e_3 = -e_3 e_2 = -c_2 e_1$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	c_1	e_3	c_1e_2	e_5	c_1e_4	$-e_7$	$-c_1e_6$
e_2	$-e_3$	c_2	$-c_2e_1$	e_6	e_7	c_2e_4	c_2e_5
e_3	$-c_1e_2$	c_2e_1	$-c_1c_2$	e_7	c_1e_6	$-c_2e_5$	$-c_1c_2e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	c_3	$-c_3e_1$	$-c_3e_2$	$-c_3e_3$
e_5	$-c_1e_4$	$-e_7$	$-c_1e_6$	c_3e_1	$-c_1c_3$	c_3e_3	$c_1c_3e_2$
e_6	e_7	$-c_2e_4$	c_2e_5	c_3e_2	$-c_3e_3$	$-c_2c_3$	$-c_2c_3e_1$
e_7	c_1e_6	$-c_2e_5$	$c_1c_2e_4$	c_3e_3	$-c_1c_3e_2$	$c_2c_3e_1$	$c_1c_2c_3$

The conjugate u^* of u is defined by

$$u^* = u_0 - \sum_{k=1}^{7} u_k e_k \tag{4}$$

and the norm $N(u)$ of u by

$$\begin{aligned} N(u)^2 &= uu^* = u^*u \\ &= u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2 - c_3u_4^2 + c_1c_3u_5^2 + c_2c_3u_6^2 - c_1c_2c_3u_7^2. \end{aligned} \tag{5}$$

Equation (5) may be rewritten as

$$N(u)^2 = \sum_{\alpha,\beta=0}^{7} \eta_{\alpha+1,\beta+1} u_\alpha u_\beta \tag{6}$$

where the $\eta_{\alpha\beta}$ define the metric

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \tag{7}$$

of an eight-dimensional Euclidean or pseudoEuclidean space.

We now analyse all the eight-dimensional algebras described by (1)-(7). Each triplet (c_1, c_2, c_3) with c_1, c_2 and $c_3 = 1$ or -1 gives rise to a metric η and to a system of seven hypercomplex numbers. The corresponding algebra $A(c_1, c_2, c_3)$ is normed or pseudonormed according to whether the metric η is Euclidean or pseudoEuclidean. The algebras $A(c_1, c_2, c_3)$ are not associative. Nevertheless, they are alternative, i.e. the associator

$$[u, v, w] = (uv)w - u(vw) \tag{8}$$

of three elements u, v and w in any $A(c_1, c_2, c_3)$ is totally antisymmetric. Furthermore, they are composition algebras, i.e.

$$N(uv)^2 = N(u)^2 N(v)^2 \tag{9}$$

for each u and v in any $A(c_1, c_2, c_3)$. The latter point, to be fully discussed in § 3, reflects the Hurwitz theorem [40] and some properties specific to Cayley-Dickson algebras of dimensions $n \leqslant 8$ [34]. The parametrised eight-dimensional algebra $A(c_1, c_2, c_3)$ generates real algebras of dimensions $n = 2$, 4 and 8 and we close this subsection by separately considering the cases $n = 8$ and $n = 4$ and 2, which cases may be associated to the various *triplets* (c_1, c_2, c_3) and to the various *generalised triplets* $(c_1, c_2, 0)$ and $(c_1, 0, 0)$, respectively.

Case $n = 2$. It corresponds to $A(c_1) \equiv A(c_1, 0, 0)$. For $c_1 = -1$, we find $A(-1) = \mathbb{C}$, the normed algebra of usual complex numbers. For $c_1 = 1$, we find $A(1) = \Omega$, the algebra of hyperbolic complex numbers [39]. This algebra admits zero-divisors since there exist elements u in Ω such that $u \neq 0$ and $N(u) = 0$. In fact,

$$N(u)^2 = u_0^2 - u_1^2 \tag{10}$$

and we can isolate in Ω the cone of all zero-divisors. Following Ilamed and Salingaros [39] we call the pseudonormed algebra Ω a singular division algebra, i.e. an almost-everywhere division algebra.

Case $n=4$. It corresponds to $A(c_1, c_2) \equiv A(c_1, c_2, 0)$. For $c_1 = c_2 = -1$, we find $A(-1, -1) = \mathbb{H}$, the normed algebra of (Hamilton) usual quaternions. For $(c_1, c_2) \neq (-1, -1)$ with c_1 and $c_2 = \pm 1$, we are led to $A(1, 1)$, $A(1, -1)$ and $A(-1, 1)$ which are all isomorphic to the algebra of hyperbolic quaternions (or Gödel quaternionic algebra or split quaternionic algebra, cf [37]). Referring to a general classification by Salingaros [39] we call $\mathbb{N}_1$ the latter pseudonormed algebra. The hyperbolic quaternionic algebra $\mathbb{N}_1$ is a singular division algebra. For instance, if u belongs to $A(-1, 1)$, then

$$N(u)^2 = u_0^2 + u_1^2 - u_2^2 - u_3^2 \tag{11}$$

and we obtain again a cone of zero-divisors.

Case $n=8$. It corresponds to $A(c_1, c_2, c_3)$. For $c_1 = c_2 = c_3 = -1$, we find $A(-1, -1, -1) = \mathbb{O}$, the normed algebra of (Cayley) usual octonions. For $(c_1, c_2, c_3) \neq (-1, -1, -1)$ with c_1, c_2 and $c_3 = \pm 1$, we obtain seven algebras isomorphic to the algebra, that we denote as $\mathbb{O}'$, of (Dickson) hyperbolic octonions. These seven pseudonormed algebras are related to eight-dimensional pseudoEuclidean spaces with metrics of the signature 0. For example, if u belongs to $A(-1, -1, 1)$, then

$$N(u)^2 = u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2. \tag{12}$$

Equation (12) shows that the hyperbolic octonionic algebra $\mathbb{O}'$ is a singular division algebra with a cone of zero-divisors.

We give a summary of the main properties for the Cayley-Dickson algebras in dimensions $n = 2$, 4 and 8 in table 2.

2.2. *Anti-involutions of the algebras and of their subalgebras*

For the purpose of defining anti-involutions of $A(c_1, c_2, c_3)$, let us first emphasise those points which are relevant for the construction of a $2n$-dimensional Cayley-Dickson algebra from a n-dimensional Cayley-Dickson algebra (cf also [35]) . We start from a (real) Cayley-Dickson algebra $A(c)$ of dimension n with $n-1$ generators $e_1, e_2, \ldots, e_{n-1}$. The argument c identifies to c_1, c_2, c_3 for $n = 8$, to c_1, c_2 for $n = 4$ and to c_1 for $n = 2$. For n arbitrary, c is a p-uple $c_1, c_2, \ldots, c_p$ where each c_i $(i = 1, 2, \ldots, p)$ stands for 1 or -1 and p is such that $2^p = n$. Then, we can construct a new Cayley-Dickson algebra $A(c, c')$, where $c' = \pm 1$, of dimension $2n$ and called the double algebra of $A(c)$. The algebra $A(c, c')$ is the set of all pairs (u, v), where u and v belong to $A(c)$, endowed with the multiplication rule

$$(u, v)(w, x) = (uw + c'x^*v,\ vw^* + xu). \tag{13}$$

(Conjugates of the type x^* for x in $A(c)$ are defined by a relation mimicking (4) in an obvious way.) Finally, with the following identifications:

$$\begin{array}{llll} A(c) \equiv \{(u, 0);\ u \in A(c)\} & & & \\ 1 \equiv (1, 0) & e_1 \equiv (e_1, 0), \ldots & e_{n-1} \equiv (e_{n-1}, 0) & \\ e_n \equiv (0, 1) & e_{n+1} \equiv (0, e_1), \ldots & e_{2n-1} \equiv (0, e_{n-1}) & \end{array} \tag{14}$$

it is possible to write

$$A(c, c') = A(c) \oplus A(c)e_n \qquad \text{with} \qquad e_n^2 = c'. \tag{15}$$

Table 2. The various Cayley-Dickson algebras $A(c_1, c_2, c_3)$.

Dimension n	Triplet (c_1, c_2, c_3)	Metric η	Algebra $A(c_1, c_2, c_3)$	Normed (N) or pseudonormed (PN) algebra	Division (D) or singular division (SD) algebra	Commutative algebra	Associative algebra	Alternative algebra	Composition algebra
2	$(-1, 0, 0)$	$(++)$	$\mathbb{C}$	N	D	yes	yes	yes	yes
2	$(1, 0, 0)$	$(+-)$	Ω	PN	SD	yes	yes	yes	yes
4	$(-1, -1, 0)$	$(++++)$	$\mathbb{H}$	N	D	no	yes	yes	yes
4	$(-1, 1, 0)$ $(1, -1, 0)$ $(1, 1, 0)$	$(++--)$ $(+-+-)$ $(+--+)$	$\mathbb{N}_1$	PN	SD	no	yes	yes	yes
8	$(-1, -1, -1)$	$(++++++++)$	$\mathbb{O}$	N	D	no	no	yes	yes
8	$(-1, -1, 1)$ $(-1, 1, -1)$ $(1, -1, -1)$ $(-1, 1, 1)$ $(1, -1, 1)$ $(1, 1, -1)$ $(1, 1, 1)$	$(++++----)$ $(++--++--)$ $(+-+-+-+-)$ $(++----++)$ $(+-+--+-+)$ $(+--++--+)$ $(+--+-++-)$	$\mathbb{O}'$	PN	SD	no	no	yes	yes

The crux of our algebraic approach amounts to constructing anti-involutions of the various Cayley-Dickson algebras $A(c_1, c_2, c_3)$, $A(c_1, c_2)$ and $A(c_1)$. We now define what we mean by an anti-involution of an arbitrary algebra A.

Definition 1. An anti-involution j of an algebra A is an involutory anti-automorphism of A, i.e. a mapping j of A into A such that $j(j(a)) = a$ and $j(ab) = j(b)j(a)$ for each a and each b in A.

Returning to the decomposition afforded by (15), we remark that the usual conjugation of $A(c)$ can be used to generate two anti-involutions of $A(c, c')$. The first one, which we call j_0, corresponds to the usual conjugation of $A(c, c')$ which may be described by

$$j_0((u, v)) = (u, v)^* = (u^*, -v). \tag{16}$$

The second one, which we call j, is defined by

$$j((u, v)) = (u^*, v). \tag{17}$$

As an example of how to manipulate these two anti-involutions, we can easily verify that

$$j((u, v)(w, x)) = (w^*u^* + c'v^*x, vw^* + xu) \tag{18}$$

and

$$j((w, x))j((u, v)) = (w^*, x)(u^*, v) \tag{19}$$

are equal.

In order to obtain other anti-involutions of a Cayley-Dickson algebra, we give a 'decomposition' process which may be thought of as the reverse of the 'composition' process described by (15). Starting with a $2m$-dimensional Cayley-Dickson algebra $A(c)$, we can find $2m-1$ m-dimensional Cayley-Dickson subalgebras $A_1, A_2, \ldots, A_{2m-1}$ of $A(c)$ such that

$$A(c) = A_k \oplus A_k e_q \qquad (k = 1, 2, \ldots, 2m-1) \tag{20}$$

where e_q is such that ue_q does not belong to A_k for each u in A_k. Equation (20) parallels (15) and, in fact, identifies with (15) for $A(c) \equiv A(c, c')$, $A_k \equiv A(c)$, $e_q \equiv e_n$ and $m \equiv n$. Then, due to (16) and (20), it is possible to find $2m$ anti-involutions $j_0, j_1, \ldots, j_{2m-1}$ of $A(c)$, j_0 being the usual conjugation of $A(c)$ and the j_k for $k = 1, 2, \ldots, 2m-1$ being the anti-involutions defined by (cf (17))

$$\begin{aligned} &j_k((u, v)) = (u^*, v) \qquad k = 1, 2, \ldots, 2m-1 \qquad \text{for } m \neq 1 \\ &j_1((u, v)) \equiv j_0((u, v)) = (u, -v) \qquad \text{for } m = 1 \end{aligned} \tag{21}$$

where u belongs to the subalgebra A_k of $A(c)$. Let us mention that the anti-involution (of the quaternionic algebra $\mathbb{H}$) considered in [43] constitutes a special case of the anti-involutions j_k $(k = 1, 2, \ldots, 2m-1)$ defined in this work.

Each anti-involution j_α (for $\alpha = 0, 1, \ldots, 2m-1$) of $A(c)$ allows us to decompose the $2m$-dimensional vector space $A(c)$ as

$$A(c) = H_\alpha \oplus M_\alpha \tag{22}$$

where the two vector subspaces M_α and H_α are such that

$$j_\alpha(H_\alpha) = -H_\alpha \qquad \text{and} \qquad j_\alpha(M_\alpha) = M_\alpha. \tag{23}$$

It is to be pointed out that M_k with $k=1,2,\ldots,2m-1$ and H_α with $\alpha=0,1,\ldots,2m-1$ are not subalgebras of $A(c)$. However, the vector space $1\oplus H_k$ for $k=1,2,\ldots,2m-1$ gives rise to the m-dimensional Cayley-Dickson algebra A_k when $m\neq 1$ and to the two-dimensional Cayley-Dickson algebra $A(c)=\mathbb{C}$ or Ω when $m=1$. Note that the $2m$-dimensional space $1\oplus H_0$ gives rise to $A(c)$. In the particular cases $2m=2$, 4 and 8, we can generate an associative algebra from a basis of the vector space H_k for $k=1,2,\ldots,2m-1$. Then, in these cases, it is possible to endow H_k with a Lie algebra structure. It is enough to take $[x,y]=xy-yx$ as Lie law, for if x and y belong to H_k, then $[x,y]$ also belongs to H_k. This Lie-admissible character of the space H_k will be used in § 3.

We now apply this to the Cayley-Dickson algebras $A(c)$ of dimensions $2m=2$, 4 and 8. At this point, it is worth recalling that if $\{e_k;\ k=1,2,\ldots,7\}$ is a system of generators of the eight-dimensional Cayley-Dickson algebra $A(c_1,c_2,c_3)$ then all the sets of generators of its quaternionic subalgebras can be deduced from the first Cayley triangle (e_1,e_2,e_3) by rotations of angle $2\pi/7$ of the heptagon $(e_1,e_3,e_4,e_5,e_6,e_2,e_7)$. More precisely, we denote H_k $(k=1,2,\ldots,7)$ the vector space deduced from (e_1,e_2,e_3) by a rotation of angle $(k-1)2\pi/7$ performed on $(e_1,e_3,e_4,e_5,e_6,e_2,e_7)$. Similarly, in the four-dimensional case, we denote H_k $(k=1,2,3)$ the vector space deduced from the point (e_1) by a rotation of angle $(k-1)2\pi/3$ performed on the triangle (e_1,e_2,e_3). Finally, in the two-dimensional case, the single H_1 corresponds to the single point (e_1). We are now in a position to examine in turn the cases $2m=2$, 4 and 8.

Case 2m = 2. In the two-dimensional case, we have either $A(c)=\mathbb{C}$ or $A(c)=\Omega$. For $A(c)=\mathbb{C}$, the space H_1 is spanned by e_1 with $e_1^2=-1$. For $A(c)=\Omega$, the space H_1 is spanned by e_1 with $e_1^2=1$.

Table 3. Generators of the possible subspaces H_k for the various Cayley-Dickson algebras $A(c_1,c_2,c_3)$.

	H_k						
$A(c_1,c_2,c_3)$	H_1	H_2	H_3	H_4	H_5	H_6	H_7
$A(c_1,0,0)$	e_1						
$A(c_1,c_2,0)$	e_1	e_2	e_3				
$A(c_1,c_2,c_3)$	e_1,e_2,e_3	e_3,e_4,e_7	e_1,e_4,e_5	e_3,e_5,e_6	e_2,e_4,e_6	e_2,e_5,e_7	e_1,e_6,e_7

Table 4. The possible algebras $1\oplus H_k$ for the various Cayley-Dickson algebras $A(c_1,c_2,c_3)$.

Dimension $2m$ of $A(c_1,c_2,c_3)$	Algebra $A(c_1,c_2,c_3)$	Possible algebra $1\oplus H_k$ $(A(c_1,c_2,c_3)=H_k\oplus M_k)$
2	$\mathbb{C}$	$\mathbb{C}$
	Ω	Ω
4	$\mathbb{H}$	$\mathbb{C}$
	$\mathbb{N}_1$	$\mathbb{C}$ or Ω
8	$\mathbb{O}$	$\mathbb{H}$
	$\mathbb{O}'$	$\mathbb{H}$ or $\mathbb{N}_1$

Case 2m = 4. In the four-dimensional case, we have either $A(c) = \mathbb{H}$ or $A(c) = \mathbb{N}_1$. First, $\mathbb{H}$ admits three (complex numbers) subalgebras $\mathbb{C}$ so that H_k is spanned by real multiples of e_k with $e_k^2 = -1$ ($k = 1$, 2 or 3). Secondly, $\mathbb{N}_1$ contains one (complex numbers) subalgebra $\mathbb{C}$ and two hyperbolic (complex numbers) subalgebras Ω. Then, H_k ($k = 1, 2, 3$) is spanned either by one of the two e_k such that $e_k^2 = 1$ or by the remaining e_k such that $e_k^2 = -1$.

Case 2m = 8. In the eight-dimensional case, we have either the usual octonionic algebra $A(c) = \mathbb{O}$ or the hyperbolic octonionic algebra $A(c) = \mathbb{O}'$. First, $\mathbb{O}$ comprises seven usual quaternionic subalgebras $\mathbb{H}$. Consequently, H_k ($k = 1, 2, \ldots, 7$) is spanned by the set of all the linear combinations of e_{p_1}, e_{p_2} and e_{p_3} such that $\{e_{p_1}, e_{p_2}, e_{p_3}\}$ is a system of generators for $\mathbb{H}$. Secondly, $\mathbb{O}'$ admits only one subalgebra of type $\mathbb{H}$ and six subalgebras isomorphic with the hyperbolic quaternionic algebra $\mathbb{N}_1$. Then, H_k ($k = 1, 2, \ldots, 7$) is spanned by the set of all the linear combinations of e_{p_1}, e_{p_2} and e_{p_3} such that $\{ep_1, e_{p_2}, e_{p_3}\}$ is a system of generators for either $\mathbb{H}$ or $\mathbb{N}_1$.

We summarise the results of this subsection in tables 3 and 4.

3. Hurwitz transformations

3.1. Definition and properties

In § 3, we restrict our attention to Cayley–Dickson algebras $A(c)$ of dimension $2m = 2$, 4 and 8. According to § 2, for fixed $2m$ we have $A(c) = H_k \oplus M_k$, where $H_k = -j_k(H_k)$ and $M_k = j_k(M_k)$, j_k being one of the $2m - 1$ anti-involutions of $A(c)$ with $k = 1, 2, \ldots, 2m - 1$ for $2m = 2$, 4 or 8. We now introduce what we call a Hurwitz transformation through the following definition.

Definition 2. The maps

$$\mathscr{H}_{\mathrm{L}}^{(k)} : A(c) \to A(c) : u \mapsto j_k(u)u \tag{24}$$

and

$$\mathscr{H}_{\mathrm{R}}^{(k)} : A(c) \to A(c) : u \mapsto uj_k(u) \tag{25}$$

are called, respectively, left and right Hurwitz transformations of $A(c)$.

We shall use the notations

$$x_{\mathrm{R}} = \mathscr{H}_{\mathrm{R}}^{(k)}(u) = uj_k(u) \tag{26}$$

and

$$x_{\mathrm{L}} = \mathscr{H}_{\mathrm{L}}^{(k)}(u) = j_k(u)u \tag{27}$$

for some fixed value of k ($k = 1, 2, \ldots, 2m - 1$). The Hurwitz transformations $x_{\mathrm{L}} = \mathscr{H}_{\mathrm{L}}^{(k)}(u)$ and $x_{\mathrm{R}} = \mathscr{H}_{\mathrm{R}}^{(k)}(u)$ satisfy properties 1–5 below. When the proof is detailed, this is done only for the right Hurwitz transformations since the proof for the left Hurwitz transformations follows the same pattern.

Property 1. The right and left Hurwitz transformations $\mathscr{K}_{\mathrm{R}}^{(k)}$ and $\mathscr{K}_{\mathrm{L}}^{(k)}$ are connected via

$$\mathscr{K}_{\mathrm{R}}^{(k)}(j_k(u)) = \mathscr{K}_{\mathrm{L}}^{(k)}(u) \tag{28}$$

and

$$\mathscr{K}_{\mathrm{L}}^{(k)}(j_k(u)) = \mathscr{K}_{\mathrm{R}}^{(k)}(u) \tag{29}$$

for each element u in the Cayley-Dickson algebra $A(c)$.

Property 2. The right and left Hurwitz transformations are transformations of magnitude 2, i.e.

$$N(x_{\mathrm{R}})^2 = N(u)^4 \tag{30}$$

and

$$N(x_{\mathrm{L}})^2 = N(u)^4. \tag{31}$$

Proof. Let us start from

$$N(x_{\mathrm{R}})^2 = x_{\mathrm{R}}x_{\mathrm{R}}^* = (uj_k(u))(uj_k(u))^* = (uj_k(u))((j_k(u))^*u^*). \tag{32}$$

We can easily verify that $(j_k(u))^* = j_k(u^*)$. Therefore, if the algebra $A(c)$ is associative (i.e. when $A(c) = \mathbb{C}$, Ω, $\mathbb{H}$ or $\mathbb{N}_1$), we can write

$$N(x_{\mathrm{R}})^2 = uj_k(u)j_k(u^*)u^* = uj_k(u^*u)u^* = (uu^*)^2 = N(u)^4. \tag{33}$$

If $A(c)$ is not associative but alternative (i.e. when $A(c) = \mathbb{O}$ or $\mathbb{O}'$), it is still possible to associate $j_k(u)$ and $j_k(u^*)$. First, we set $u = u_0 + v$ and $j_k(u) = w$ with u_0 a real number and v a pure (hyperbolic) octonion. Then,

$$\begin{aligned} N(x_{\mathrm{R}})^2 &= ((u_0+v)w)(w^*(u_0-v)) \\ &= (u_0w)(w^*u_0) - (u_0w)(w^*v) + (vw)(w^*u_0) - (vw)(w^*v). \end{aligned} \tag{34}$$

Second, the desired result follows by applying to (34) two properties (see [44]), namely, the identity $u(u^*v) = (vu)u^*$, which holds for any u and v in an alternative algebra endowed with a conjugation, and the so-called Moufang identity $u(vw)u = (uv)(wu)$, which holds for any u, v and w in an arbitrary alternative algebra.

Property 3. We have

$$\mathscr{K}_{\mathrm{R}}^{(k)}(A(c)) \subseteq M_k^+ \qquad \text{and} \qquad \mathscr{K}_{\mathrm{L}}^{(k)}(A(c)) \subseteq M_k^+ \tag{35}$$

where M_k^+ is the subset of M_k containing elements of squared norm positive or null only.

Proof. For any u in $A(c)$, we obtain

$$j_k(\mathscr{K}_{\mathrm{R}}^{(k)}(u)) = \mathscr{K}_{\mathrm{R}}^{(k)}(u). \tag{36}$$

Consequently, $\mathscr{K}_{\mathrm{R}}^{(k)}(u)$ belongs to M_k. The proof is completed by noting that, according to property 2, $\mathscr{K}_{\mathrm{R}}^{(k)}(u)$ has a squared norm positive or null.

Property 4. The manifold $G_k = \exp(H_k)$ is a Lie group.

Proof. It follows immediately from the Lie-admissible structure of H_k which is outlined in § 2.

From the definition of H_k, we remark that each element of G_k is either a (hyperbolic) complex number of norm ± 1 when $2m = 2$ and 4 or a (hyperbolic) quaternion of norm ± 1 when $2m = 8$.

Property 5. The coset $G_k/\mathbb{Z}_2$ is the kernel of the Hurwitz transformations $\mathscr{K}_{\mathrm{R}}^{(k)}$ and $\mathscr{K}_{\mathrm{L}}^{(k)}$.

Proof. Let v be an element of G_k. According to our preceding remark, v is of norm ± 1. Furthermore, we have

$$j_k(v) = v^* \tag{37}$$

because v is either a (hyperbolic) complex number or a (hyperbolic) quaternion. Let us now calculate $\mathscr{K}_{\mathrm{R}}^{(k)}(uv)$ for an arbitrary element u in $A(c)$. We get

$$\mathscr{K}_{\mathrm{R}}^{(k)}(uv) = (uv)(v^* j_k(u)). \tag{38}$$

When the dimension of $A(c)$ is $2m = 2$ or 4, the right-hand side of (38) is easily seen to be $uj_k(u)$. The same result may be reached when the dimension of $A(c)$ is $2m = 8$. To prove the latter assertion it is convenient: first, to decompose u as $u = (w, x)$, where x and w are, in the notations of (13)-(21), two elements of the algebra $1 \oplus H_k$ and second, to realise that $vwv^* = w$. As a partial conclusion, we have $\mathscr{K}_{\mathrm{R}}^{(k)}(uv) = uj_k(u)$ for u in $A(c)$ and v in G_k. Furthermore, each v in G_k can be multiplied by -1 without any change of $\mathscr{K}_{\mathrm{R}}^{(k)}(uv) = uj_k(u)$ and the kernel of $\mathscr{K}_{\mathrm{R}}^{(k)}$ is hence $G_k/\mathbb{Z}_2$.

Table 5 lists all the kernels $G_k/\mathbb{Z}_2$ corresponding to $2m = 2$, 4 and 8. For $2m = 2$, the space H_k is generated either by a pure complex number or by a pure hyperbolic complex number, so that $\exp(H_k)$ identifies to either U(1) or $\mathbb{R}\oplus\mathbb{R}$ and $G_k/\mathbb{Z}_2$ to either SO(2) or $\mathrm{SO}_0(1,1)$, respectively. For $2m = 4$, the same reasoning leads to the kernel SO(2) in the quaternionic case and to the kernel SO(2) or $\mathrm{SO}_0(1,1)$ in the hyperbolic quaternionic case. For $2m = 8$, we first consider the octonionic case. Then, we know that H_k is generated by a pure quaternion. Therefore, $\exp(H_k)$ identifies to SU(2) and the coset $G_k/\mathbb{Z}_2$ to SO(3). Secondly, in the hyperbolic octonionic case, H_k is generated either by a pure quaternion or by a pure hyperbolic quaternion, so that $\exp(H_k)$ identifies to either SU(2) or SU(1, 1) and $G_k/\mathbb{Z}_2$ to either SO(3) or $\mathrm{SO}_0(1,2)$, respectively.

3.2. Hurwitz transformations and Clifford algebras

In this subsection, we establish contact between the Hurwitz transformations and Clifford algebras. We shall here again work directly in eight dimensions since the

Table 5. The possible kernels $G_k/\mathbb{Z}_2$ for the various Cayley-Dickson algebras $A(c_1, c_2, c_3)$.

Dimension $2m$ of $A(c_1, c_2, c_3)$	Algebra $A(c_1, c_2, c_3)$	Kernel $G_k/\mathbb{Z}_2$
2	$\mathbb{C}$	SO(2)
	Ω	$\mathrm{SO}_0(1,1)$
4	$\mathbb{H}$	SO(2)
	$\mathbb{N}_1$	SO(2) or $\mathrm{SO}_0(1,1)$
8	$\mathbb{O}$	SO(3)
	$\mathbb{O}'$	SO(3) or $\mathrm{SO}_0(1,2)$

$(2m=4)$- and $(2m=2)$-dimensional cases straightforwardly follow from the $(2m=8)$-dimensional case.

From the generators $e_1, e_2, \ldots, e_7$ of the eight-dimensional Cayley-Dickson algebra $A(c_1, c_2, c_3)$, we define the following eight-component vector:

$$e=\begin{pmatrix} 1 \\ e_1 \\ \vdots \\ e_7 \end{pmatrix}. \tag{39}$$

Then the product $e_k e$ produces a real matrix Γ_k of order eight through

$$e_k e = \Gamma_k e \qquad (k=1,2,\ldots,7). \tag{40}$$

In appendix 1, we give the 8×8 matrices Γ_k for $k=1,2,\ldots,7$ in explicit form. From the definition of the Γ_k, we readily obtain the property

$$\tilde{\Gamma}_k = -\eta\Gamma_k\eta \qquad (k=1,2,\ldots,7). \tag{41}$$

Furthermore, we have the following preliminary lemma.

Lemma. The set $\{\Gamma_k;\ k=1,2,\ldots,7\}$ generates the Clifford algebra $\mathscr{C}(p,q)$ of degree $p+q=7$ with $-p+q$ being the signature of the metric g.

Proof. It is sufficient to combine (1) and (40). This leads to

$$\Gamma_k\Gamma_l+\Gamma_l\Gamma_k=-2g_{kl}\mathbb{1}_8 \qquad (k \text{ and } l=1,2,\ldots,7) \tag{42}$$

which almost completes the proof. (The $2m$-dimensional unit matrix is denoted as $\mathbb{1}_{2m}$.) The $(2m=4)$- and $(2m=2)$-dimensional cases may be obtained from evident restrictions of (39)-(42).

Table 6 identifies the Clifford algebras of degree $2m-1$ generated by $\{\Gamma_k;\ k=1,2,\ldots,2m-1\}$ for $2m=2$, 4 and 8. The notations $\mathscr{C}(p,q)$, $\mathbb{N}_r$ and $\mathbb{R}(s)$ in table 6 refer to the work of Salingaros [45]. Note that $\mathbb{N}_1$ is isomorphic to $\mathbb{R}(2)$, $\mathbb{R}(s)$ being the algebra on $\mathbb{R}$ of $s\times s$ real matrices.

We are now in a position to translate the product of two arbitrary elements of $A(c_1, c_2, c_3)$ in terms of the Clifford matrices Γ_k. To each element u in $A(c_1, c_2, c_3)$,

Table 6. The Clifford algebras $\mathscr{C}(p,q)$ associated with the various Cayley-Dickson algebras $A(c_1, c_2, c_3)$.

Dimension $2m$ of $A(c_1,c_2,c_3)$	Triplet (c_1,c_2,c_3)	Clifford algebra $\mathscr{C}(p,q)$
2	$(c_1,0,0)=(-1,0,0)$	$\mathscr{C}(0,1)\simeq\mathbb{C}$
	$(c_1,0,0)=(1,0,0)$	$\mathscr{C}(1,0)\simeq\Omega$
4	$(c_1,c_2,0)=(-1,-1,0)$	$\mathscr{C}(0,3)\simeq\mathbb{H}\oplus\mathbb{H}$
	$(c_1,c_2,0)\neq(-1,-1,0)$	$\mathscr{C}(2,1)\simeq\mathbb{N}_1\oplus\mathbb{N}_1$
8	$(c_1,c_2,c_3)=(-1,-1,-1)$	$\mathscr{C}(0,7)\simeq\mathbb{R}(8)\oplus\mathbb{R}(8)$
	$(c_1,c_2,c_3)\neq(-1,-1,-1)$	$\mathscr{C}(4,3)\simeq\mathbb{R}(8)\oplus\mathbb{R}(8)$

cf (3), we associate the eight-dimensional (real) vector

$$\boldsymbol{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_7 \end{pmatrix}. \tag{43}$$

Then (3) can be rewritten

$$u = \tilde{\boldsymbol{u}}\boldsymbol{e} = \tilde{\boldsymbol{e}}\boldsymbol{u}. \tag{44}$$

Further, the product uv for u and v belonging to the algebra $A(c_1, c_2, c_3)$ is, in matrix form,

$$uv = \left(u_0 + \sum_{k=1}^{7} u_k e_k \right) \tilde{\boldsymbol{e}}\boldsymbol{v} \tag{45}$$

or, equivalently,

$$uv = \tilde{\boldsymbol{e}} \left(u_0 \mathbb{1}_8 + \sum_{k=1}^{7} u_k \tilde{\Gamma}_k \right) \boldsymbol{v}. \tag{46}$$

By introducing the 8×8 matrix

$$A(\boldsymbol{u}) = u_0 \mathbb{1}_8 + \sum_{k=1}^{7} u_k \tilde{\Gamma}_k \tag{47}$$

the product uv can be finally transcribed as

$$uv = \tilde{\boldsymbol{e}} A(\boldsymbol{u}) \boldsymbol{v}. \tag{48}$$

The explicit expression of $A(\boldsymbol{u})$ is obtained by putting the matrices Γ_k of appendix 1 into (47). This yields

$$A(\boldsymbol{u}) = \begin{pmatrix}
u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 & c_3u_4 & -c_1c_3u_5 & -c_2c_3u_6 & c_1c_2c_3u_7 \\
u_1 & u_0 & c_2u_3 & -c_2u_2 & c_3u_5 & -c_3u_4 & c_2c_3u_7 & -c_2c_3u_6 \\
u_2 & -c_1u_3 & u_0 & c_1u_1 & c_3u_6 & -c_1c_3u_7 & -c_3u_4 & c_1c_3u_5 \\
u_3 & -u_2 & u_1 & u_0 & c_3u_7 & -c_3u_6 & c_3u_5 & -c_3u_4 \\
u_4 & -c_1u_5 & -c_2u_6 & c_1c_2u_7 & u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 \\
u_5 & -u_4 & -c_2u_7 & c_2u_6 & u_1 & u_0 & -c_2u_3 & c_2u_2 \\
u_6 & c_1u_7 & -u_4 & -c_1u_5 & u_2 & c_1u_3 & u_0 & -c_1u_1 \\
u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & -u_1 & u_0
\end{pmatrix}. \tag{49}$$

This matrix $A(\boldsymbol{u})$ satisfies the following properties:

$$\tilde{A}(\boldsymbol{u}) = \eta A(\boldsymbol{u}^*) \eta \tag{50}$$

and

$$\tilde{A}(\boldsymbol{u}) \eta A(\boldsymbol{u}) = (\tilde{\boldsymbol{u}} \eta \boldsymbol{u}) \eta. \tag{51}$$

Equations (50), where $\boldsymbol{u}^*$ is the matrix representative of $u^* = \tilde{\boldsymbol{e}}\boldsymbol{u}^*$, and (51) can be proved by replacing $A(\boldsymbol{u})$ by its expression (see (47)) in terms of the matrices Γ_k. We shall refer to (51) as a generalised Hurwitz property since it constitutes an extension of a property established by Hurwitz in the special cases $c_1 = c_2 - 1 = c_3 - 1 = -1$, $c_1 = c_2 = c_3 - 1 = -1$ and $c_1 = c_2 = c_3 = -1$ in connection with the famous factorisation problem of the sum of 2, 4 and 8 squares, respectively [40]. In this vein, (51) provides us with the solution of the following generalised (Hurwitz) problem. Given $\tilde{\boldsymbol{u}} = (u_0 u_1 \ldots u_7)$ and $\tilde{\boldsymbol{v}} = (v_0 v_1 \ldots v_7)$, find $\tilde{\boldsymbol{w}} = (w_0 w_1 \ldots w_7)$ such that

$$\tilde{\boldsymbol{w}}\eta\boldsymbol{w} = (\tilde{\boldsymbol{u}}\eta\boldsymbol{u})(\tilde{\boldsymbol{v}}\eta\boldsymbol{v}). \tag{52}$$

The solution is (up to matrices pseudo-orthogonal with respect to the metric η)

$$\boldsymbol{w} = A(\boldsymbol{u})\boldsymbol{v} \tag{53}$$

where $A(\boldsymbol{u})$ satisfies (51) and may be taken as given by (49). (By up to pseudo-orthogonal matrices we mean that we may replace, in (53), the matrix $A(\boldsymbol{u})\boldsymbol{v}$ by $RA(\boldsymbol{u})S\boldsymbol{v}$, where R and S are matrices pseudo-orthogonal with respect to the metric η.) Note that (52) and (53) constitute indeed a rewriting of (9) with $uv = w$. When converted in arbitrary dimension, the latter generalised problem can be seen to admit solutions only in dimensions $2m = 8$, 4 and 2. The solutions in dimensions 4 and 2 may be derived by taking the four- and two-dimensional restrictions of (49)-(53) respectively. (The matrices $A(\boldsymbol{u})$ and η in two or four dimensions are simple restrictions of the corresponding matrices in eight dimensions.)

Let us now introduce the partial conjugation matrices $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7$ via

$$\varepsilon_k = \operatorname{diag}(1, \varepsilon_1^{(k)}, \ldots, \varepsilon_7^{(k)}) \qquad (k = 1, 2, \ldots, 7) \tag{54}$$

where $\varepsilon_p^{(k)}$ is defined as follows. We take $\varepsilon_p^{(k)} = -1$ if p belongs to $\{p_1, p_2, p_3\}$ such that $\{e_{p_1}, e_{p_2}, e_{p_3}\}$ is a basis for the vector space H_k (cf § 2) and $\varepsilon_p^{(k)} = 1$ if not. (From § 2, it is clear that the various possible $\{p_1, p_2, p_3\}$ are: $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$ and $\{3, 5, 6\}$.) Then, we are able to realise the anti-involution j_k (for $k = 1, 2, \ldots, 7$) by noting that

$$j_k(u) = \tilde{\boldsymbol{e}}\varepsilon_k\boldsymbol{u}. \tag{55}$$

It is clear that the vector $\varepsilon_k\boldsymbol{u}$ corresponds to an element of $A(c_1, c_2, c_3)$ that is a partial conjugate of u (with three changes of sign among $u_0, u_1, \ldots, u_7$). In the realisation afforded by (44), (48) and (55), the right Hurwitz transformation $\mathscr{H}_{\mathrm{R}}^{(k)}$ is described by

$$\mathscr{H}_{\mathrm{R}}^{(k)}: A(c_1, c_2, c_3) \to M_k^-: \tilde{\boldsymbol{e}}\boldsymbol{u} \mapsto \tilde{\boldsymbol{e}}A(\boldsymbol{u})\varepsilon_k\boldsymbol{u}. \tag{56}$$

Similarly, for the left Hurwitz transformation $\mathscr{H}_{\mathrm{L}}^{(k)}$ we have

$$\mathscr{H}_{\mathrm{L}}^{(k)}: A(c_1, c_2, c_3) \to M_k^-: \tilde{\boldsymbol{e}}\boldsymbol{u} \mapsto \tilde{\boldsymbol{e}}A(\varepsilon_k\boldsymbol{u})\boldsymbol{u}. \tag{57}$$

It is convenient to use the shorthand notation

$$x = \mathscr{H}_{\mathrm{R}}^{(k)}(u) \qquad \text{or} \qquad x = \mathscr{H}_{\mathrm{L}}^{(k)}(u) \tag{58}$$

for some fixed k, so that the eight-dimensional (real) vector $\boldsymbol{x}$ associated to $x (= \tilde{\boldsymbol{e}}\boldsymbol{x})$ is

$$\boldsymbol{x} = A(\boldsymbol{u})\varepsilon_k\boldsymbol{u} \qquad \text{or} \qquad \boldsymbol{x} = A(\varepsilon_k\boldsymbol{u})\boldsymbol{u} \tag{59}$$

respectively. We remark that (30) and (31), and therefore property 2, can be transcribed as $\tilde{\boldsymbol{x}}\eta\boldsymbol{x} = (\tilde{\boldsymbol{u}}\eta\boldsymbol{u})^2$ with the notation of (58) and (59). We shall continue with illustrative examples concerning right Hurwitz transformations (since similar results apply to both right and left Hurwitz transformations) in the cases $2m = 8$, 4 and 2. The 'inverses' of the right Hurwitz transformations displayed below are relegated to appendix 2.

Case 2m = 8. We choose the right Hurwitz transformation $x = \mathscr{K}_R^{(1)}(u)$ with H_1 being the vector space spanned by e_1, e_2 and e_3 (cf table 3). Therefore, we have

$$\boldsymbol{x} = A(\boldsymbol{u})\varepsilon_1\boldsymbol{u} \qquad \text{with} \qquad \varepsilon_1 = \operatorname{diag}(1, -1, -1, -1, 1, 1, 1, 1). \tag{60}$$

A simple development of (60) leads to the following explicit expression:

$$\begin{aligned}
&x_0 = u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2 + c_3u_4^2 - c_1c_3u_5^2 - c_2c_3u_6^2 + c_1c_2c_3u_7^2\\
&x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0\\
&x_4 = 2(u_0u_4 + c_1u_1u_5 + c_2u_2u_6 - c_1c_2u_3u_7)\\
&x_5 = 2(u_0u_5 + u_1u_4 + c_2u_2u_7 - c_2u_3u_6)\\
&x_6 = 2(u_0u_6 - c_1u_1u_7 + u_2u_4 + c_1u_3u_5)\\
&x_7 = 2(u_0u_7 - u_1u_6 + u_2u_5 + u_3u_4)
\end{aligned} \tag{61}$$

for the eight-dimensional transformation $x = \mathscr{K}_R^{(1)}(u)$. From (61), we can easily verify that

$$\begin{aligned}
&x_0^2 - c_3x_4^2 + c_1c_3x_5^2 + c_2c_3x_6^2 - c_1c_2c_3x_7^2\\
&\qquad = (u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2 - c_3u_4^2 + c_1c_3u_5^2 + c_2c_3u_6^2 - c_1c_2c_3u_7^2)^2
\end{aligned} \tag{62}$$

which turns out to be the explicit form of (30) for $x_R \equiv x = \mathscr{K}_R^{(k)}(u)$ with u in $A(c_1, c_2, c_3)$.

Case 2m = 4. The restriction of (49) to the four-dimensional space yields

$$A(\boldsymbol{u}) = \begin{pmatrix} u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3\\ u_1 & u_0 & c_2u_3 & -c_2u_2\\ u_2 & -c_1u_3 & u_0 & c_1u_1\\ u_3 & -u_2 & u_1 & u_0 \end{pmatrix}. \tag{63}$$

We choose H_3, the vector space generated by e_3 (cf table 3). The right Hurwitz transformation $x = \mathscr{K}_R^{(3)}(u)$ corresponding to H_3 is described by the relations

$$\begin{aligned}
&x_0 = u_0^2 + c_1u_1^2 + c_2u_2^2 + c_1c_2u_3^2\\
&x_1 = 2(u_0u_1 + c_2u_2u_3)\\
&x_2 = 2(u_0u_2 - c_1u_1u_3)\\
&x_3 = 0
\end{aligned} \tag{64}$$

from which we get

$$x_0^2 - c_1x_1^2 - c_2x_2^2 = (u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2)^2. \tag{65}$$

In the special case where $c_1 = c_2 = -1$, (64) corresponds to the usual KS transformation up to a relabelling of the indices α in u_α ($\alpha = 0, 1, 3, 4$) and of the indices i in x_i ($i = 0, 1, 2$). In this special case, our derivation of the Hurwitz transformations by means of anti-involutions parallels the well known approach of the KS transformation from the map of $\mathbb{H}$ onto $\mathbb{H}$ defined by (cf also [43])

$$u_0 + u_1e_1 + u_2e_2 + u_3e_3 \mapsto (u_0 + u_1e_1 + u_2e_2 + u_3e_3)(u_0 + u_1e_1 + u_2e_2 - u_3e_3). \tag{66}$$

In the special case where $c_1 = c_2 = 1$, (64) corresponds to the transformation used by Iwai [42] to reduce Hamiltonian systems with two degrees of freedom.

Case 2m = 2. The restriction of (49) and (61) to the two-dimensional case leads to the matrix

$$A(\boldsymbol{u}) = \begin{pmatrix} u_0 & c_1 u_1 \\ u_1 & u_0 \end{pmatrix} \tag{67}$$

and to the right Hurwitz transformation $x = \mathscr{K}_{\mathrm{R}}^{(1)}(u)$ given by

$$x_0 = u_0^2 - c_1 u_1^2 \qquad x_1 = 0 \tag{68}$$

respectively. The latter transformation is (like the two preceding ones) a transformation of magnitude two since

$$x_0^2 - c_1 x_1^2 = (u_0^2 - c_1 u_1^2)^2. \tag{69}$$

We close this subsection with a word of comment concerning the situation where the anti-involutions j_k for $k = 1, 2, \ldots, 2m-1$ are replaced by the usual conjugation j_0. There is then no difference between right and left Hurwitz transformations. Indeed, for fixed $2m$, the Hurwitz transformations $\mathscr{K}_{\mathrm{L}}^{(0)} = \mathscr{K}_{\mathrm{R}}^{(0)} \equiv \mathscr{K}^{(0)}$ correspond to applications from $\mathbb{R}^{2m}$ onto $\mathbb{R}^+$ or $\mathbb{R}$ according to whether the metric g is Euclidean or pseudo-Euclidean. (It should be noted that the transformation $\mathscr{K}^{(0)}$ may be defined for $2m$ arbitrary and that the transformations $\mathscr{K}_{\mathrm{R}}^{(k)}$ and $\mathscr{K}_{\mathrm{L}}^{(k)}$ $(k = 1, 2, \ldots, 2m-1)$ may be defined with the maximum of properties for $2m = 2$, 4 and 8 only.) In the case $2m = 8$, the Hurwitz transformation $x = \mathscr{K}^{(0)}(u)$ is obtained by substituting

$$\varepsilon_0 = \mathrm{diag}(1, -1, -1, -1, -1, -1, -1, -1) \tag{70}$$

for ε_k in (56)-(59). This gives

$$x_0 = \tilde{\boldsymbol{u}}\eta\boldsymbol{u} \qquad x_1 = x_2 = \ldots = x_7 = 0. \tag{71}$$

The cases $2m = 4$ and 2 correspond to evident restrictions of (71). In the case $2m = 2$, it is clear that the Hurwitz transformations $\mathscr{K}_{\mathrm{L}}^{(1)}$ and $\mathscr{K}_{\mathrm{R}}^{(1)}$ identify to $\mathscr{K}^{(0)}$.

4. QuasiHurwitz transformations

4.1. Definition and properties

We start here with an arbitrary Cayley-Dickson algebra $A(c)$ of dimension $2m$. Then we introduce the following definition and list two immediate properties.

Definition 3. The application

$$\mathscr{L}: A(c) \to A(c): u \mapsto x = \mathscr{L}(u) = u^2 \tag{72}$$

is called a quasiHurwitz transformation of $A(c)$.

Property 6. The relation

$$N(x)^2 = N(u)^4 \tag{73}$$

holds for any element u in $A(c)$.

Proof. It follows from the application of a theorem by Artin [46] to

$$N(x)^2 = N(\mathscr{L}(u))^2 = u^2(u^2)^* = (uu)(u^*u^*). \tag{74}$$

Property 7. The quasiHurwitz transformation $\mathscr{L}$ has a discrete kernel of type $\mathbb{Z}_2$.

This property is evident from definition 3. It is thus possible to consider the transformation $\mathscr{L}$ as a map from $A(c)$ onto $A(c)/\mathbb{Z}_2$.

4.2. *QuasiHurwitz transformations in matrix form*

Let the $2m$-dimensional (real) vector

$$\boldsymbol{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{2m-1} \end{pmatrix} \tag{75}$$

and the $2m \times 2m$ matrix

$$A(\boldsymbol{u}) = u_0 \mathbb{1}_{2m} + \sum_{k=1}^{2m-1} u_k \tilde{\Gamma}_k \tag{76}$$

be the $2m$-dimensional generalisations of (43) and (47) respectively. Equations (75) and (76) allow us to write $x = \mathscr{L}(u)$ in matrix form as

$$\boldsymbol{x} = A(\boldsymbol{u})\boldsymbol{u} \tag{77}$$

and (77) is the matrix expression of the quasiHurwitz transformation $\mathscr{L}(u) = u^2$ of the $2m$-dimensional algebra $A(c)$. (Equation (77) has a form quasisimilar to the one of (60), a fact that is at the origin of the nomenclature 'quasiHurwitz'.) We now focus our attention on some special cases corresponding to particular values of the dimension $2m$.

Case $2m = 2$. Equation (77) leads to

$$x_0 = u_0^2 + c_1 u_1^2 \qquad x_1 = 2u_0 u_1. \tag{78}$$

In the compact case $c_1 = -1$, the algebra $A(-1)$ is $\mathbb{C}$ and the quasiHurwitz transformation $x = \mathscr{L}(u)$ is nothing but the well known LC transformation. Therefore, by putting

$$\omega = x_0 + \mathrm{i}x_1 \qquad z = u_0 + \mathrm{i}u_1 \qquad \mathrm{i}^2 = c_1 = -1 \tag{79}$$

the transformation $\mathscr{L}$ turns out to be the conformal map

$$\mathbb{C} \to \mathbb{C} : z \mapsto \omega = z^2. \tag{80}$$

It is obvious that

$$\frac{\partial x_0}{\partial u_0} = \frac{\partial x_1}{\partial u_1} \qquad \frac{\partial x_0}{\partial u_1} = -\frac{\partial x_1}{\partial u_0} \tag{81}$$

which express the analycity conditions of Cauchy and Riemann, and that

$$\mathrm{d}x_0^2 + \mathrm{d}x_1^2 = 4(u_0^2 + u_1^2)(\mathrm{d}u_0^2 + \mathrm{d}u_1^2) \tag{82}$$

which reflects the conformal nature of the LC transformation. By imposing $u_0^2 + u_1^2 = 1$, it follows from property 6 that $x_0^2 + x_1^2 = 1$ and, according to property 7, the corresponding quasiHurwitz transformation $\mathscr{L}$ may be seen as a map from S^1 onto $S^1/\mathbb{Z}_2$, the real projective space $\mathbb{R}P^1$.

In the non-compact case $c_1 = 1$, the algebra $A(1)$ is Ω and the quasiHurwitz transformation $\mathscr{L}$ may be thought of as the map

$$\Omega \to \Omega : z \mapsto \omega = z^2 \tag{83}$$

with

$$\omega = x_0 + ix_1 \qquad z = u_0 + iu_1 \qquad i^2 = c_1 = 1. \tag{84}$$

We can readily check that

$$\frac{\partial x_0}{\partial u_0} = \frac{\partial x_1}{\partial u_1} \qquad \frac{\partial x_0}{\partial u_1} = \frac{\partial x_1}{\partial u_0} \tag{85}$$

which are referred to as hyperbolic analycity conditions by Laurentiev and Chabat [38], and that

$$\mathrm{d}x_0^2 - \mathrm{d}x_1^2 = 4(u_0^2 - u_1^2)(\mathrm{d}u_0^2 - \mathrm{d}u_1^2) \tag{86}$$

which reflects the so-called (cf [38]) hyperbolic conformal nature of the map $\mathscr{L}$.

Case 2m = 4. Equation (77) yields

$$\begin{aligned} x_0 &= u_0^2 + c_1u_1^2 + c_2u_2^2 - c_1c_2u_3^2 \\ x_1 &= 2u_0u_1 \qquad x_2 = 2u_0u_2 \qquad x_3 = 2u_0u_3. \end{aligned} \tag{87}$$

In the compact case $c_1 = c_2 = -1$, the quasiHurwitz transformation $\mathscr{L}$ is related to the real projective space $\mathbb{R}P^3$. As a matter of fact, by taking $u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1$, we have $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, so that $\mathscr{L}$ can be considered as a map from S^3 onto $S^3/\mathbb{Z}_2$ which is precisely $\mathbb{R}P^3$.

Case 2m = 8. The quasiHurwitz transformation $x = \mathscr{L}(u)$ is

$$\begin{aligned} x_0 &= u_0^2 + c_1u_1^2 + c_2u_2^2 - c_1c_2u_3^2 + c_3u_4^2 - c_1c_3u_5^2 - c_2c_3u_6^2 + c_1c_2c_3u_7^2 \\ x_1 &= 2u_0u_1 \qquad x_2 = 2u_0u_2 \qquad x_3 = 2u_0u_3 \qquad x_4 = 2u_0u_4 \\ x_5 &= 2u_0u_5 \qquad x_6 = 2u_0u_6 \qquad x_7 = 2u_0u_7. \end{aligned} \tag{88}$$

In the compact case $c_1 = c_2 = c_3 = -1$, repeating what we have just done in the two- and four-dimensional cases, the quasiHurwitz transformation $\mathscr{L}$ can be written as a map from S^7 onto $S^7/\mathbb{Z}_2 = \mathbb{R}P^7$.

We are now ready to briefly discuss the general case where $2m$ is arbitrary. In this case, the quasiHurwitz transformation $x = \mathscr{L}(u)$ becomes

$$x_0 = 2u_0^2 - \tilde{\boldsymbol{u}}\eta\boldsymbol{u} \qquad x_k = 2u_0u_k \qquad (k = 1, 2, \ldots, 2m-1) \tag{89}$$

where η is the $2m$-dimensional extension of the metric defined by (2) and (7). Equation (89) gives back (78), (87) and (88) as particular cases. In the notation of (89), (73) and therefore property 6 can be translated as $\tilde{\boldsymbol{x}}\eta\boldsymbol{x} = (\tilde{\boldsymbol{u}}\eta\boldsymbol{u})^2$. Note that the reverse of (89) is formally given by

$$\begin{aligned} u_0 &= \pm[x_0 \pm (\tilde{\boldsymbol{x}}\eta\boldsymbol{x})^{1/2}]^{1/2}/\sqrt{2} \\ u_k &= x_k/(2u_0) \qquad (k = 1, 2, \ldots, 2m-1). \end{aligned} \tag{90}$$

Furthermore, the line element $\mathrm{d}s^2 = \mathrm{d}\tilde{\boldsymbol{x}}\,\eta\,\mathrm{d}\boldsymbol{x}$ can be shown to satisfy

$$\mathrm{d}\tilde{\boldsymbol{x}}\,\eta\,\mathrm{d}\boldsymbol{x} = 4\,\mathrm{d}u_0^2\,\tilde{\boldsymbol{u}}\eta\boldsymbol{u} + 4u_0^2\,\mathrm{d}\tilde{\boldsymbol{u}}\,g\,\mathrm{d}\boldsymbol{u} + 4(\mathrm{d}\tilde{\boldsymbol{u}}\,g\boldsymbol{u})^2 \tag{91}$$

where g is defined by the extension of (2) to the $2m$-dimensional case. It is remarkable that (91) reduces to the conformal expression (see (82)) and to the hyperbolic conformal expression (see (86)) in the particular case $2m=2$. Finally, in the $2m$-dimensional compact case (i.e. for $c_1=c_2=\ldots=c_p=-1$ with $2m=2^p$), the quasiHurwitz transformation $\mathscr{L}$ appears to be a map from S^{2m-1} onto $S^{2m-1}/\mathbb{Z}_2$ which identifies to the real projective space $\mathbb{R}P^{2m-1}$.

5. Geometrical aspects of the Hurwitz transformations

In this section we suppose that u is an element of $A(c_1, c_2, c_3)$ such that $N(u)^2=1$ and examine the Hurwitz transformations $x=\mathscr{H}_{\mathrm{L}}^{(k)}(u)$ or $\mathscr{H}_{\mathrm{R}}^{(k)}(u)$ for the two-, four- and eight-dimensional cases in the compact and non-compact versions. The compact version yields Hopf fibrations (on spheres) and the non-compact one what we call pseudoHopf fibrations (on hyperboloids).

5.1. The two-dimensional case

In this case, u belongs to $\mathbb{C}=A(-1)$ or $\Omega=A(1)$ and $N(u)^2=1$ means that u is on the real sphere S^1 (of equation $u_0^2+u_1^2=1$) or on the hyperbola $H^1(1,1)$ (of equation $u_0^2-u_1^2=1$), respectively. Consequently, the Hurwitz transformation $x=\mathscr{H}_{\mathrm{L}}^{(1)}(u)=\mathscr{H}_{\mathrm{R}}^{(1)}(u)$ induces a trivial map from S^1 or $H^1(1,1)$ onto $\{1\}$ according to whether u is an element of $\mathbb{C}$ or Ω.

5.2. The four-dimensional case

If u belongs to $\mathbb{H}=A(-1,-1)$, then $N(u)^2=1$ means that u stands on the real sphere S^3. The Hurwitz transformations $x=\mathscr{H}_{\mathrm{L}}^{(k)}(u)$ and $\mathscr{H}_{\mathrm{R}}^{(k)}(u)$ for $k=1$, 2 or 3 satisfy

$$\left(\sum_{\alpha=0}^{3} u_\alpha^2\right)^2=\sum_{\alpha=0}^{3} x_\alpha^2=1 \tag{92}$$

with one of the x_α vanishing. Then each of the latter transformations (which all correspond to the KS transformation) describes the Hopf fibration on spheres $S^3\to S^2$ of fibre SO(2) (that is homeomorphic to the real sphere S^1), a known result as far as the KS transformation is concerned.

If u belongs to $\mathbb{N}_1=A(c_1, c_2)$ with $c_i=\pm 1$ (for $i=1$ and 2) and $(c_1, c_2)\neq(-1,-1)$, then the condition

$$N(u)^2=u_0^2-c_1u_1^2-c_2u_2^2+c_1c_2u_3^2=1 \tag{93}$$

means that u is on a single-sheeted hyperboloid which we denote as $H^3(2,2)$. We restrict ourselves to $(c_1, c_2)=(-1,1)$, a choice which does not induce any loss of generality. Thus, the equation of $H^3(2,2)$ is

$$u_0^2+u_1^2-u_2^2-u_3^2=1. \tag{94}$$

Following the results of § 3, we can foresee two types of Hurwitz transformations, namely either (i) transformations with compact fibre or (ii) transformations with non-compact fibre.

(i) An example of the first type of Hurwitz transformation is $x=\mathscr{H}_{\mathrm{R}}^{(1)}(u)$ which gives

$$(u_0^2+u_1^2-u_2^2-u_3^2)^2=x_0^2-x_2^2-x_3^2=1. \tag{95}$$

This transformation describes the pseudoHopf fibration on hyperboloids $H^3(2,2) \to H^2(1,2)_0$ of fibre $SO(2) \approx S^1$, where $H^2(1,2)_0$ is the upper sheet of the two-sheeted hyperboloid $H^2(1,2)$ of equation

$$x_0^2 - x_2^2 - x_3^2 = 1. \tag{96}$$

(It is easy to verify that $\mathscr{H}_R^{(1)}(H^3(2,2))$ is $H^2(1,2)_0$ rather than $H^2(1,2)$ because $x_0 = \Sigma_{\alpha=0}^3 u_\alpha^2$ remains positive.) We note that this fibration is inherent to the work of Iwai [42] on the reduction, by an S^1 action, of Hamiltonian systems with two degrees of freedom.

(ii) We consider now the second type of Hurwitz transformation in the four-dimensional non-compact case with $(c_1, c_2) = (-1, 1)$. An example of such a type of transformation is $x = \mathscr{H}_R^{(3)}(u)$ which gives

$$(u_0^2 + u_1^2 - u_2^2 - u_3^2)^2 = x_0^2 + x_1^2 - x_2^2 = 1. \tag{97}$$

This Hurwitz transformation describes a new pseudoHopf fibration on hyperboloids, namely the fibration $H^3(2,2) \to H^2(2,1)$ of fibre $SO_0(1,1)$ (that is homeomorphic to the real line $\mathbb{R}$), where $H^2(2,1)$ is the single-sheeted hyperboloid of equation

$$x_0^2 + x_1^2 - x_2^2 = 1. \tag{98}$$

Similar results may be derived for the doublets $(c_1, c_2) = (1, -1)$ and $(1, 1)$ in the cases (i) and (ii).

Topologically, $H^3(2,2)$ is homeomorphic to $\mathbb{R}^2 \times S^1$. Therefore, the fibrations (i) $H^3(2,2) \to H^2(1,2)_0$ of fibre S^1 and (ii) $H^3(2,2) \to H^2(2,1)$ of fibre $\mathbb{R}$ are trivial because $H^2(1,2)_0$ and $H^2(2,1)$ are homeomorphic to $\mathbb{R}^2$ and $\mathbb{R} \times S^1$, respectively.

5.3. *The eight-dimensional case*

The Hurwitz transformations $x = \mathscr{H}_L^{(k)}(u)$ and $\mathscr{H}_R^{(k)}(u)$ with $k = 1, 2, \ldots$ or 7 for u in $\mathbb{O} = A(-1,-1,-1)$ such that $N(u)^2 = 1$ satisfy

$$\left(\sum_{\alpha=0}^{7} u_\alpha^2\right)^2 = \sum_{\alpha=0}^{7} x_\alpha^2 = 1 \tag{99}$$

with three of the x_α vanishing. Then, each of the latter transformations describes the well known Hopf fibration on spheres $S^7 \to S^4$ of fibre $SO(3)$ (that is homeomorphic to the real sphere S^3).

If u belongs to $\mathbb{O}' = A(c_1, c_2, c_3)$ with $c_i = \pm 1$ (for $i = 1, 2$ and 3) and $(c_1, c_2, c_3) \neq (-1,-1,-1)$, then the condition $N(u)^2 = 1$ (see (5)) means that u lies on a single-sheeted hyperboloid which we denote as $H^7(4,4)$. Without any loss of generality, we take $(c_1, c_2, c_3) = (-1,-1,1)$. Hence, the equation of $H^7(4,4)$ is

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2 = 1. \tag{100}$$

We foresee two types of Hurwitz transformations: (i) Hurwitz transformations with compact fibre and (ii) Hurwitz transformations with non-compact fibre.

(i) An example of the first type of Hurwitz transformation is $x = \mathscr{H}_R^{(1)}(u)$ which yields

$$(u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2)^2 = x_0^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 = 1. \tag{101}$$

This transformation leads to a new pseudoHopf fibration on hyperboloids, namely $H^7(4,4) \to H^4(1,4)_0$ of fibre $SO(3) \approx S^3$, where $H^4(1,4)_0$ is the upper sheet of the two-sheeted hyperboloid $H^4(1,4)$ of equation

$$x_0^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 = 1. \tag{102}$$

(It is clear that $\mathscr{H}_{\mathrm{R}}^{(1)}(H^7(4,4)) = H^4(1,4)_0$ and not $H^4(1,4)$ since $x_0 = \Sigma_{\alpha=0}^{7} u_\alpha^2$ is positive.)

(ii) Let us now consider the second type of Hurwitz transformation in the eight-dimensional non-compact case with $(c_1, c_2, c_3) = (-1, -1, 1)$. An example of such a type of transformation is $x = \mathscr{H}_{\mathrm{R}}^{(3)}(u)$ which gives

$$(u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2)^2 = x_0^2 + x_2^2 + x_3^2 - x_6^2 - x_7^2 = 1. \qquad (103)$$

This Hurwitz transformation describes a new pseudoHopf fibration on hyperboloids, namely $H^7(4,4) \to H^4(3,2)$ of fibre $\mathrm{SO}_0(1,2)$ (that is homeomorphic to $\mathbb{R}^2 \times S^1$, a fact readily understood if we remember that $\mathrm{SO}(1,2) \simeq \mathrm{SU}(1,1)/\mathbb{Z}_2$), where $H^4(3,2)$ is the hyperboloid of equation

$$x_0^2 + x_2^2 + x_3^2 - x_6^2 - x_7^2 = 1. \qquad (104)$$

Of course, similar results apply, both for (i) and (ii), to the six other triplets $(c_1, c_2, c_3) \neq (-1, -1, -1)$ and $(-1, -1, 1)$.

Topologically, $H^7(4,4)$ is homeomorphic to $\mathbb{R}^4 \times S^3$ while $H^4(1,4)_0$ and $H^4(3,2)$ are homeomorphic to $\mathbb{R}^4$ and $\mathbb{R}^2 \times S^2$, respectively. Consequently, (i) the pseudoHopf fibration $H^7(4,4) \to H^4(1,4)_0$ of fibre S^3 is a trivial fibration while (ii) the pseudoHopf fibration $H^7(4,4) \to H^4(3,2)$ of fibre $\mathbb{R}^2 \times S^1$ is a non-trivial fibration (cf the non-triviality of the classical Hopf fibration $S^3 \to S^2$ of fibre S^1).

6. Differential aspects of the Hurwitz transformations

The aim of this section is to provide some differential expressions for the Hurwitz transformations. We shall deal with right Hurwitz transformations since the left Hurwitz transformations may be treated in the same way. In each of the cases $2m = 8$, 4 and 2, we shall consider a typical Hurwitz transformation.

6.1. *The two-dimensional case*

There is only one Hurwitz transformation, namely $\mathscr{H}_{\mathrm{R}}^{(1)}$. According to § 3, such a transformation may be described by the 2×2 matrix $A(\boldsymbol{u})$ of (67) via the relation $\boldsymbol{x} = A(\boldsymbol{u})\varepsilon_1\boldsymbol{u}$. Then, it is straightforward to derive five properties.

Property 8. We have

$$\begin{pmatrix} \mathrm{d}x_0 \\ \omega \end{pmatrix} = 2A(\boldsymbol{u})\varepsilon_1 \begin{pmatrix} \mathrm{d}u_0 \\ \mathrm{d}u_1 \end{pmatrix} \qquad (105)$$

where $\omega = 2(u_1\,\mathrm{d}u_0 - u_0\,\mathrm{d}u_1)$ is a 1-form rather than a total differential. Because the transformation $\mathscr{H}_{\mathrm{R}}^{(1)}$ maps $\mathbb{R}^2$ onto $\mathbb{R}$ or $\mathbb{R}^+$, it is possible to require $\omega = 0$. This leads to the following property.

Property 9. We have

$$\mathrm{d}x_0^2 = 4r(\mathrm{d}u_0^2 - c_1\,\mathrm{d}u_1^2) \qquad (106)$$

where $r = N(u)^2 = u_0^2 - c_1 u_1^2$.

We can now derive the transformation law of the partial derivation operators $(\partial/\partial u_\alpha)$ $(\alpha = 0, 1)$ under $\mathscr{H}_{\mathrm{R}}^{(1)}$.

Property 10. We get

$$\begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ \dfrac{1}{2r}X \end{pmatrix} = \frac{1}{2r}\eta A(\boldsymbol{u})\varepsilon_1\eta \begin{pmatrix} \dfrac{\partial}{\partial u_0} \\ \dfrac{\partial}{\partial u_1} \end{pmatrix} \tag{107a}$$

where $\eta = \mathrm{diag}(1, -c_1)$ and

$$X = -c_1 u_1 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_1}. \tag{107b}$$

The operator X can be interpreted as a vector field defined in the basis $\{\partial/\partial u_0, \partial/\partial u_1\}$ of a tangent space at the point (u_0, u_1).

If we introduce the action of a 1-form du_α on the vector $\partial/\partial u_\beta$ by $du_\alpha(\partial/\partial u_\beta) = \delta(\alpha, \beta)$, we obtain the next property.

Property 11. We have $\omega[(1/2r)X] = 1$.

As a corollary of property 10, it is possible to express the transformation law of second-order (elliptic or hyperbolic) differential operators under $\mathscr{K}_{\mathrm{R}}^{(1)}$. By defining ∇_u by $\tilde{\nabla}_u = [(\partial/\partial u_0)(\partial/\partial u_1)]$, we may set out the last property.

Property 12. We have

$$\frac{\partial^2}{\partial x_0^2} + \frac{1}{x_0}\frac{\partial}{\partial x_0} = \frac{1}{4r}\tilde{\nabla}_u \eta \nabla_u + \frac{1}{4r^2}c_1 X^2. \tag{108}$$

When $c_1 = -1$, (108) takes the particular form

$$\frac{\partial^2}{\partial x_0^2} + \frac{1}{x_0}\frac{\partial}{\partial x_0} = \frac{1}{4r}\Delta_u - \frac{1}{4r^2}X^2 \tag{109a}$$

where Δ_u stands for the two-dimensional Laplacian and

$$X = u_1 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_1} \tag{109b}$$

is the infinitesimal generator of a compact group SO(2). Equation (109*a*) was equally well derived by Kibler and Négadi [47].

When $c_1 = 1$, (108) specialises to

$$\frac{\partial^2}{\partial x_0^2} + \frac{1}{x_0}\frac{\partial}{\partial x_0} = \frac{1}{4r}\Box_u + \frac{1}{4r^2}X^2 \tag{110a}$$

where $\Box_u$ is the D'Alembertian

$$\Box_u = \frac{\partial^2}{\partial u_0^2} - \frac{\partial^2}{\partial u_1^2} \tag{110b}$$

and

$$X = -u_1 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_1} \tag{110c}$$

the infinitesimal generator of a non-compact group SO(1, 1). It is to be realised that the left-hand side of (110*a*) cannot be defined on the 'light' cone $u_0^2 - u_1^2 = 0$.

6.2. The four-dimensional case

We consider the Hurwitz transformation $\mathscr{H}_{\mathrm{R}}^{(3)}(u)$ described by the 4×4 matrix $A(\boldsymbol{u})$ of (63). The infinitesimal version of $\boldsymbol{x}=A(\boldsymbol{u})\varepsilon_3\boldsymbol{u}$ may be understood through the following property.

Property 13. We have

$$\begin{pmatrix} \mathrm{d}x_0 \\ \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \omega \end{pmatrix} = 2A(\boldsymbol{u})\varepsilon_3 \begin{pmatrix} \mathrm{d}u_0 \\ \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_4 \end{pmatrix} \tag{111}$$

where the 1-form $\omega = 2(u_3\,\mathrm{d}u_0 - u_0\,\mathrm{d}u_3 + u_1\,\mathrm{d}u_2 - u_2\,\mathrm{d}u_1)$ is not a total differential. The transformation $\mathscr{H}_{\mathrm{R}}^{(3)}$ maps $\mathbb{R}^4$ onto $\mathbb{R}^3$ or $\mathbb{R}^+\times\mathbb{R}^2$ so that it is possible to demand that $\omega=0$. This yields (112*a*) below.

Property 14. We get

$$\mathrm{d}x_0^2 - c_1\,\mathrm{d}x_1^2 - c_2\,\mathrm{d}x_2^2 = 4r(\mathrm{d}u_0^2 - c_1\,\mathrm{d}u_1^2 - c_2\,\mathrm{d}u_2^2 + c_1c_2\,\mathrm{d}u_3^2) \tag{112a}$$

where

$$r = N(u)^2 = u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2. \tag{112b}$$

The transformation law of the partial derivation operators $\partial/\partial u_\alpha$ ($\alpha = 0, 1, 2, 3$) under $\mathscr{H}_{\mathrm{R}}^{(3)}$ is given by a property (property 15) which turns out to be the analogue of property 10.

Property 15. We have

$$\begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{1}{2r}X \end{pmatrix} = \frac{1}{2r}\eta A(\boldsymbol{u})\varepsilon_3\eta \begin{pmatrix} \dfrac{\partial}{\partial u_0} \\ \dfrac{\partial}{\partial u_1} \\ \dfrac{\partial}{\partial u_2} \\ \dfrac{\partial}{\partial u_3} \end{pmatrix} \tag{113a}$$

where η is the metric $\eta = \mathrm{diag}(1, -c_1, -c_2, c_1c_2)$ and X a vector field

$$X = c_1c_2u_3\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_3} + c_2u_2\frac{\partial}{\partial u_1} - c_1u_1\frac{\partial}{\partial u_2}. \tag{113b}$$

The action of the 1-form ω on the vector field $(1/2r)X$ is given by a property analogous to property 11.

At this point, we encounter a difference between the four-dimensional and the two-dimensional cases. Indeed, the following property, which concerns the transformation law of second-order differential operators under $\mathscr{K}_{\mathrm{R}}^{(3)}$, should be compared with property 12.

Property 16. We have

$$\frac{\partial^2}{\partial x_0^2} - c_1 \frac{\partial^2}{\partial x_1^2} - c_2 \frac{\partial^2}{\partial x_2^2} = \frac{1}{4r} \tilde{\nabla}_u \eta \nabla_u - \frac{1}{4r^2} c_1 c_2 X^2 \tag{114a}$$

where ∇_u is defined by

$$\tilde{\nabla}_u = \left(\frac{\partial}{\partial u_0} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \frac{\partial}{\partial u_3} \right). \tag{114b}$$

When $c_1 = c_2 = -1$, (114a) may be particularised as

$$\Delta_x = \frac{1}{4r} \Delta_u - \frac{1}{4r^2} X^2 \tag{115a}$$

where Δ_x and Δ_u are the three-dimensional and four-dimensional Laplacians in the variables x_i $(i = 0, 1, 2)$ and u_α $(\alpha = 0, 1, 2, 3)$ respectively. Further, the operator

$$X = u_3 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_3} + u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} \tag{115b}$$

happens to be the infinitesimal generator of a group of type SO(2). This may be easily seen with the following parametrisation:

$$\begin{aligned} u_0 &= \sqrt{r} \cos \frac{\phi + \psi}{2} \cos \frac{\theta}{2} & u_1 &= \sqrt{r} \cos \frac{\phi - \psi}{2} \sin \frac{\theta}{2} \\ u_2 &= \sqrt{r} \sin \frac{\phi - \psi}{2} \sin \frac{\theta}{2} & u_3 &= \sqrt{r} \sin \frac{\phi + \psi}{2} \cos \frac{\theta}{2} \end{aligned} \tag{116}$$

which leads to $X = -2\, \partial/\partial\psi$. (Note that the latter choice of coordinates gives $x_0 = r \cos \theta$, $x_1 = r \sin \theta \cos \phi$ and $x_2 = r \sin \theta \sin \phi$, cf [8] and [48].) We close the four-dimensional compact case $c_1 = c_2 = -1$ with three remarks. First, (115a) was obtained by Kibler and Négadi [48]. Second, the four- and three-dimensional volume elements are connected through $\mathrm{d}\mu(\mathbb{R}^3) = (4N(u)^2/\pi)\, \mathrm{d}\mu(\mathbb{R}^4)$; such a connecting formula appears to be of considerable importance for physical applications of the $\mathbb{R}^4 \to \mathbb{R}^3$ Hurwitz transformation as we shall show in a forthcoming paper. Third, it was mentioned by Vivarelli [49] that the constraint $\omega = 0$ is related to the Souriau quantisation of the symplectic manifold S^2.

When $c_1 = c_2 = 1$, (114a) yields (for $r = u_0^2 - u_1^2 - u_2^2 + u_3^2 \neq 0$)

$$\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} = \frac{1}{4r} \left(\frac{\partial^2}{\partial u_0^2} - \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} \right) - \frac{1}{4r^2} X^2 \tag{117a}$$

where

$$X = u_3 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_3} + u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} \tag{117b}$$

is the infinitesimal generator of a group of type SO(2). The latter point may be seen with the parametrisation

$$\begin{aligned} u_0 &= \sqrt{r}\cos\tfrac{1}{2}(\phi+\psi)\cosh\tfrac{1}{2}\theta \\ u_1 &= \sqrt{r}\sin\tfrac{1}{2}(\phi-\psi)\sinh\tfrac{1}{2}\theta \\ u_2 &= \sqrt{r}\cos\tfrac{1}{2}(\phi-\psi)\sinh\tfrac{1}{2}\theta \\ u_3 &= \sqrt{r}\sin\tfrac{1}{2}(\phi+\psi)\cosh\tfrac{1}{2}\theta \end{aligned} \quad\Rightarrow\quad \begin{aligned} X &= -2\,\partial/\partial\psi \\ x_0 &= r\cosh\theta \\ x_1 &= r\sinh\theta\sin\phi \\ x_2 &= r\sinh\theta\cos\phi. \end{aligned} \tag{118}$$

When $c_1 = \mp 1$ and $c_2 = \pm 1$, (114*a*) leads to (for $r = u_0^2 \pm u_1^2 \mp u_2^2 - u_3^2 \neq 0$)

$$\frac{\partial^2}{\partial x_0^2} \pm \frac{\partial^2}{\partial x_1^2} \mp \frac{\partial^2}{\partial x_2^2} = \frac{1}{4r}\left(\frac{\partial^2}{\partial u_0^2} \pm \frac{\partial^2}{\partial u_1^2} \mp \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2}\right) + \frac{1}{4r^2}X_\pm^2 \tag{119a}$$

where

$$X_\pm = -u_3\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_3} \pm u_2\frac{\partial}{\partial u_1} \pm u_1\frac{\partial}{\partial u_2} \tag{119b}$$

is the infinitesimal generator of a group of type SO(1, 1). This can be seen from the parametrisations

$$\begin{aligned} u_0 &= \sqrt{r}\cosh\tfrac{1}{2}(\phi-\psi)\cos\tfrac{1}{2}\theta \\ u_1 &= \sqrt{r}\cosh\tfrac{1}{2}(\phi+\psi)\sin\tfrac{1}{2}\theta \\ u_2 &= \sqrt{r}\sinh\tfrac{1}{2}(\phi+\psi)\sin\tfrac{1}{2}\theta \\ u_3 &= \sqrt{r}\sinh\tfrac{1}{2}(\phi-\psi)\cos\tfrac{1}{2}\theta \end{aligned} \quad\Rightarrow\quad \begin{aligned} X_+ &= 2\,\partial/\partial\psi \\ x_0 &= r\cos\theta \\ x_1 &= r\sin\theta\cosh\phi \\ x_2 &= r\sin\theta\sinh\phi \end{aligned} \tag{120}$$

and

$$\begin{aligned} u_0 &= \sqrt{r}\cosh\tfrac{1}{2}(\phi+\psi)\cos\tfrac{1}{2}\theta \\ u_1 &= \sqrt{r}\sinh\tfrac{1}{2}(\psi-\phi)\sin\tfrac{1}{2}\theta \\ u_2 &= \sqrt{r}\cosh\tfrac{1}{2}(\psi-\phi)\sin\tfrac{1}{2}\theta \\ u_3 &= \sqrt{r}\sinh\tfrac{1}{2}(\phi+\psi)\cos\tfrac{1}{2}\theta \end{aligned} \quad\Rightarrow\quad \begin{aligned} X_- &= -2\,\partial/\partial\psi \\ x_0 &= r\cos\theta \\ x_1 &= r\sin\theta\sinh\phi \\ x_2 &= r\sin\theta\cosh\phi. \end{aligned} \tag{121}$$

6.3. The eight-dimensional case

We choose the Hurwitz transformation $x = \mathscr{K}_{\mathrm{R}}^{(1)}(u)$ discussed in § 3 by means of the 8×8 matrix $A(\boldsymbol{u})$ of (49) and we list the five following properties.

Property 17. We have

$$\begin{pmatrix} \mathrm{d}x_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \mathrm{d}x_4 \\ \mathrm{d}x_5 \\ \mathrm{d}x_6 \\ \mathrm{d}x_7 \end{pmatrix} = 2A(\boldsymbol{u})\varepsilon_1 \begin{pmatrix} \mathrm{d}u_0 \\ \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_3 \\ \mathrm{d}u_4 \\ \mathrm{d}u_5 \\ \mathrm{d}u_6 \\ \mathrm{d}u_7 \end{pmatrix} \tag{122}$$

where the 1-forms ω_1, ω_2 and ω_3, which are not total differentials, are

$$\begin{aligned} \omega_1 = 2(&u_1\,\mathrm{d}u_0 - u_0\,\mathrm{d}u_1 + c_2u_2\,\mathrm{d}u_3 - c_2u_3\,\mathrm{d}u_2 + c_3u_5\,\mathrm{d}u_4 \\ &- c_3u_4\,\mathrm{d}u_5 + c_2c_3u_7\,\mathrm{d}u_6 - c_2c_3u_6\,\mathrm{d}u_7) \end{aligned}$$

$$\omega_2 = 2(u_2\,du_0 - u_0\,du_2 + c_1u_3\,du_1 - c_1u_1\,du_3 + c_3u_6\,du_4 - c_3u_4\,du_6 + c_1c_3u_5\,du_7 - c_1c_3u_7\,du_5) \tag{123}$$

$$\omega_3 = 2(u_3\,du_0 - u_0\,du_3 + u_2\,du_1 - u_1\,du_2 + c_3u_7\,du_4 - c_3u_4\,du_7 + c_3u_5\,du_6 - c_3u_6\,du_5).$$

In view of the fact that the transformation $x = \mathscr{K}_{\mathbb{R}}^{(1)}(u)$ described by $\boldsymbol{x} = A(\boldsymbol{u})\varepsilon_1\boldsymbol{u}$ is an $\mathbb{R}^8 \to \mathbb{R}^5$ or $\mathbb{R}^+ \times \mathbb{R}^4$ surjection, we may assume that $\omega_1 = \omega_2 = \omega_3 = 0$.

Property 18. The constraints $\omega_1 = \omega_2 = \omega_3 = 0$ make it possible to obtain

$$\begin{aligned} dx_0^2 - c_3\,dx_4^2 + c_1c_3\,dx_5^2 + c_2c_3\,dx_6^2 - c_1c_2c_3\,dx_7^2 &= 4r(du_0^2 - c_1\,du_1^2 - c_2\,du_2^2 + c_1c_2\,du_3^2 - c_3\,du_4^2 \\ &\quad + c_1c_3\,du_5^2 + c_2c_3\,du_6^2 - c_1c_2c_3\,du_7^2) \end{aligned} \tag{124}$$

where $r = N(u)^2$ is given by (5).

Property 19. We have

$$\begin{pmatrix} \dfrac{\partial}{\partial x_0} \\ \dfrac{1}{2r}X_1 \\ \dfrac{1}{2r}X_2 \\ \dfrac{1}{2r}X_3 \\ \dfrac{\partial}{\partial x_4} \\ \dfrac{\partial}{\partial x_5} \\ \dfrac{\partial}{\partial x_6} \\ \dfrac{\partial}{\partial x_7} \end{pmatrix} = \frac{1}{2r}\eta A(\boldsymbol{u})\varepsilon_1\eta \begin{pmatrix} \dfrac{\partial}{\partial u_0} \\ \dfrac{\partial}{\partial u_1} \\ \dfrac{\partial}{\partial u_2} \\ \dfrac{\partial}{\partial u_3} \\ \dfrac{\partial}{\partial u_4} \\ \dfrac{\partial}{\partial u_5} \\ \dfrac{\partial}{\partial u_6} \\ \dfrac{\partial}{\partial u_7} \end{pmatrix} \tag{125}$$

where the metric η is defined by (2) and (7), and the vector fields X_1, X_2 and X_3 are

$$\begin{aligned} X_1 &= -c_1u_1\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_1} - c_1u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3} + c_1u_5\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_5} - c_1u_7\frac{\partial}{\partial u_6} - u_6\frac{\partial}{\partial u_7} \\ X_2 &= -c_2u_2\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_2} + c_2u_3\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_3} + c_2u_6\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_6} + c_2u_7\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_7} \\ X_3 &= c_1c_2u_3\frac{\partial}{\partial u_0} - u_0\frac{\partial}{\partial u_3} - c_2u_2\frac{\partial}{\partial u_1} + c_1u_1\frac{\partial}{\partial u_2} - c_1c_2u_7\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_7} - c_2u_6\frac{\partial}{\partial u_5} + c_1u_5\frac{\partial}{\partial u_6}. \end{aligned} \tag{126}$$

It is easy to check the commutation relations

$$[X_1, X_2] = 2X_3 \qquad [X_2, X_3] = -2c_2X_1 \qquad [X_3, X_1] = -2c_1X_2. \tag{127}$$

Therefore, the set $\{X_1, X_2, X_3\}$ generates the Lie algebra of SO(3) or SO(1, 2) according to whether $(c_1, c_2) = (-1, -1)$ or $(c_1, c_2) \neq (-1, -1)$.

Property 20. The action of the 1-form ω_j on the vector field $(1/2r)X_k$ is given by

$$\omega_j\left(\frac{1}{2r}X_k\right)=\delta(j,k) \qquad (j,k=1,2,3). \tag{128}$$

Property 21. Elliptic and hyperbolic operators are connected for $r\neq 0$ through

$$\frac{\partial^2}{\partial x_0^2}-c_3\frac{\partial^2}{\partial x_4^2}+c_1c_3\frac{\partial^2}{\partial x_5^2}+c_2c_3\frac{\partial^2}{\partial x_6^2}-c_1c_2c_3\frac{\partial^2}{\partial x_7^2}$$
$$=\frac{1}{4r}\tilde{\nabla}_u\eta\nabla_u+\frac{1}{4r^2}c_1X_1^2+\frac{1}{4r^2}c_2X_2^2-\frac{1}{4r^2}c_1c_2X_3^2 \tag{129}$$

where ∇_u is defined as in the two- and four-dimensional cases. Equation (129) may be worked out for the various possible choices of the triplets (c_1, c_2, c_3). Let us just mention that in the compact case $c_1=c_2=c_3=-1$, (129) becomes

$$\Delta_x=\frac{1}{4r}\Delta_u-\frac{1}{4r^2}X_1^2-\frac{1}{4r^2}X_2^2-\frac{1}{4r^2}X_3^2 \tag{130}$$

where Δ_x and Δ_u are the five-dimensional and the eight-dimensional Laplacians in the variables x_i $(i=0,4,5,6,7)$ and u_α $(\alpha=0,1,\ldots,7)$, respectively. The seven remaining (non-compact) cases are left to the reader as an exercise.

We close this section with the following remark. In the situation where $r=1$, the Hurwitz transformations considered in this section lead, according to § 5, to Hopf and pseudoHopf fibrations. Then, the operators of type X, which occur in (108), (114) and (129), define vector fields tangential to the fibre of the above mentioned fibre bundles.

Work is currently in progress by one of the authors (DL) on the use of differential properties of non-compact Hurwitz transformations and of the corresponding pseudoHopf fibrations for solving field equations of several non-linear sigma models on curved non-compact spaces.

7. Concluding remarks and applications

The aim of this section is threefold. First, we sum in § 7.1 the main mathematical results of this work. Second, in § 7.2, we deal with some physical applications of the formalism developed in the present paper. We limit ourselves in § 7.2 to enunciating results since applications will be the object of separate publications. Third, we examine in § 7.3, on the basis of preliminary results already obtained, some other possible applications of our formalism.

7.1. The main results

(*a*) We have generalised three $\mathbb{R}^{2m}\to\mathbb{R}^{2m-n}$ quadratic transformations useful in theoretical physics, namely, the Levi-Civita transformation [1], the Kustaanheimo–Stiefel transformation [6, 7] and a transformation recently introduced by Iwai [42]. This has led us to compact and non-compact (i) $\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ quasiHurwitz transformations, with $2m$ arbitrary, which comprise the (compact) LC transformation for $2m=2$ and (ii)

$\mathbb{R}^{2m} \to \mathbb{R}^{2m-n}$ Hurwitz transformations, with $(2m, 2m-n) = (2, 1)$, $(4, 3)$ and $(8, 5)$, which comprise the (compact) KS transformation and the (non-compact) Iwai transformation for $(2m, 2m-n) = (4, 3)$. All these transformations have been obtained in a unified algebraic framework based on the use of Cayley-Dickson and Clifford algebras.

(*b*) We have investigated the quasiHurwitz and Hurwitz transformations from a geometrical viewpoint. Among the most important results is the demonstration that the compact Hurwitz transformations are related to some of the famous Hopf fibrations on spheres and the non-compact Hurwitz transformations to new fibrations, i.e. fibrations on hyperboloids. More specifically, the $\mathbb{R}^{2m} \to \mathbb{R}^{2m-n}$ Hurwitz transformations may be geometrically classified as follows. (i) For $(2m, 2m-n) = (8, 5)$, we have three types of Hurwitz transformations: two types with a compact fibre S^3 and one type with a non-compact fibre $\mathbb{R}^2 \times S^1$. (ii) For $(2m, 2m-n) = (4, 3)$, similar results apply. We obtain two types of Hurwitz transformations with a compact fibre S^1 and one type of Hurwitz transformations with a non-compact fibre $\mathbb{R}$. (iii) For the limiting case $(2m, 2m-n) = (2, 1)$, there are only two distinct Hurwitz transformations: one with a compact fibre S^1 and another with a non-compact fibre $\mathbb{R}$.

(*c*) We have studied the transformation properties of various quantities (as line elements, gradient operators and Laplacians or d'Alembertians) under quasiHurwitz and Hurwitz transformations. Such a study constitutes an indispensable preliminary for passing to physical applications. In particular, a detailed examination of the contents of § 6 yields the following result: the different $\mathbb{R}^{2m} \to \mathbb{R}^{2m-n}$ Hurwitz transformations connect generalised Laplace operators, in $\mathbb{R}^{2m-n}$ and $\mathbb{R}^{2m}$, invariant under Lie groups (say G_{2m-n} and G_{2m} respectively) of type orthogonal or pseudo-orthogonal. For instance, the possible couples $[G_{2m-n}, G_{2m}]$ are for the non-trivial cases $2m = 4$ and 8: (i) $[O(5), O(8)]$, $[O(1, 4), O(4, 4)]$ and $[O(3, 2), O(4, 4)]$ when $(2m, 2m-n) = (8, 5)$ and (ii) $[O(3), O(4)]$, $[O(1, 2), O(2, 2)]$ and $[O(2, 1), O(2, 2)]$ when $(2m, 2m-n) = (4, 3)$. The latter result dictates the general philosophy for applying Hurwitz transformations to classical or quantum theory. More precisely, a given $\mathbb{R}^{2m} \to \mathbb{R}^{2m-n}$ Hurwitz transformation allows us to convert a problem involving a symmetry group G_{2m-n} (and possibly an invariance group G'_{2m-n}) into a problem in higher dimension and involving a symmetry group G_{2m} (and possibly an invariance group G'_{2m}).

(*d*) As a by-product of our algebraic approach to non-bijective quadratic transformations, we have obtained a non-compact extension of the Hurwitz theorem [40] on the product of two sums each containing $2m = 2$, 4 or 8 squared integers. The central result is that the non-compact extension can be performed only if the considered sums involve metrics of the type $(++++----)$ for $2m = 8$, $(++--)$ for $2m = 4$ and $(+-)$ for $2m = 2$.

7.2. *Applications*

We show here the interest in our algebraic and geometric formalism for (i) the search for spectrum generating algebras and dynamical invariance algebras for simple quantum mechanical systems and (ii) non-linear sigma models on curved spaces. We only show the main results because a complete treatment can be found in separate publications [52-54].

7.2.1. Invariance and non-invariance algebras. We begin with an example. The dynamical invariance algebra for the bound states of the hydrogen atom and the spectrum generating algebra for the whole spectrum of the hydrogen atom are known to be so(4) (see [55]) and so(4, 2) (see [56]), respectively. Both algebras (or the associated covering Lie groups) are obtained, in the standard treatments, by raising the number of degrees of freedom. We thus pass from the symmetry group SO(3) to the dynamical invariance group SO(4) and from SO(4) to the spectrum generating group SO(4, 2). In the framework of our formalism, the corresponding Lie algebras may be obtained very quickly by a restriction process as follows. The Hurwitz transformation $\mathscr{K}_{\mathbb{R}}^{(1)}$ with $c_1 = c_2 = c_3 - 1 = -1$ allow the transformation of the Schrödinger equation for the hydrogen atom in $\mathbb{R}^3$ into the Schrödinger equation for an isotropic harmonic oscillator in $\mathbb{R}^4$ subjected to a constraint. First, let us consider the case of the discrete levels for the hydrogen atom. Then the oscillator is an ordinary one (i.e. with attractive potential), the dynamical invariance algebra of which is su(4). By introducing in su(4) the constraint $X = 0$, cf (114) with $c_1 = c_2 = -1$, we end up with a *Lie algebra under constraint* which is isomorphic to so(4), the dynamical invariance algebra for the discrete spectrum of the hydrogen atom. Second, if we introduce the vanishing vector field $X = 0$ in sp(8, $\mathbb{R}$), the non-invariance algebra for a four-dimensional isotropic oscillator, we obtain another Lie algebra under constraint which turns out to be isomorphic to so(4, 2), the spectrum generating algebra for the three-dimensional hydrogen atom.

Lie algebras under constraint(s) may be obtained for each of the Hurwitz transformations associated to $2m = 8$, 4 and 2. It is sufficient to introduce $X_1 = X_2 = X_3 = 0$ (see (125)) in sp(16, $\mathbb{R}$) for $2m = 8$; $X = 0$ (see (113a)) in sp(8, $\mathbb{R}$) for $2m = 4$; and $X = 0$ (see (107a)) in sp(4, $\mathbb{R}$) for $2m = 2$. The main results, to be proved at length elsewhere [53], are the following (L_1 stands for a Lie algebra under constraint(s)). (i) For $(2m, 2m - n) = (8, 5)$, we obtain $L_1 = \mathrm{so}(6, 2)$ for all transformations with an S^3 fibre and $L_1 = \mathrm{so}(4, 4)$ for all transformations with an $\mathbb{R}^2 \times S^1$ fibre. (ii) For $(2m, 2m - n) = (4, 3)$, we obtain $L_1 = \mathrm{so}(4, 2)$ for all transformations with an S^1 fibre and $L_1 = \mathrm{so}(3, 3)$ for all transformations with an $\mathbb{R}$ fibre. (iii) For $(2m, 2m - n) = (2, 1)$, we obtain $L_1 = \mathrm{so}(2, 1)$ for the transformation with an S^1 fibre and $L_1 = \mathrm{so}(2, 1)$ for the transformation with an $\mathbb{R}$ fibre. From a mathematical point of view, the remarkable result is that there is a one-to-one correspondence between Lie algebras under constraint(s) and types of fibre. Therefore, for a given doublet $(2m, 2m - n)$, there are only two types of Lie algebras under constraint(s) corresponding either to a compact fibre or a non-compact fibre. The latter result may be rationalised, *a posteriori*, if we realise that the fibrations of § 5 with compact (non-compact) fibres are associated with compact (non-compact) constraint operators.

As an immediate consequence of result (ii), we may foresee that the spectrum generating algebra of a hydrogen atom in $\mathbb{R}^8$ is so(6, 2). Another immediate application concerns the Hartmann potential $V = -a/r + b/(r \sin \theta)^2$ with $a > 0$ and $b > 0$. This potential has recently received a great deal of attention [52, 57] both in a Schrödinger partial differential equation picture and a Feynman path integral picture. The relevant chain of Lie algebras for the Hartmann potential problem is clearly $\mathrm{sp}(8, \mathbb{R}) \supset \mathrm{su}(4) \supset \mathrm{so}(4) \supset \mathrm{so}(2) \times \mathrm{so}(2)$ if use is made of the compact Hurwitz transformation for $2m = 4$. The dynamical invariance algebra for the three-dimensional Hartmann potential then corresponds to the Lie algebra so(4) under the constraint $X = 0$ with $c_1 = c_2 = c_3 - 1 = -1$. The resulting Lie algebra under constraint is simply su(2), a result fully discussed in [53].

7.2.2. Sigma models. Let M and N be two Riemannian manifolds. Then, a smooth map $f: M \to N$ is said to be a solution of the sigma model defined on M, with values on N, if and only if it is a *harmonic map* [58] from M to N. We use $f: M \underset{F}{\rightarrow} N$ to denote the fibrations of § 5.

As a first result, we can prove that each fibration $f: M \underset{F}{\rightarrow} N$, of fibre F, is harmonic. Therefore, f is a solution of the sigma model defined on M and with values on N. For the compact cases, this result is not new. As a matter of fact, it is well known that the classical Hopf fibrations define harmonic maps, a property recently reinterpreted in the language of sigma models by Fujii (see [41]). For the non-compact cases, our result happens to be new and gives explicit realisations of harmonic maps between hyperboloids. For example, the fibrations $H^3(2,2) \rightarrow H^2(1,2)_0$ and $H^7(4,4) \rightarrow H^4(1,4)_0$ provide explicit solutions for the sigma models $H^1(\mathbb{K}) \rightarrow \mathbb{K}PS^1$, where $H^1(\mathbb{K})$ is the 1-hyperboloid on $\mathbb{K}^2$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ and $\mathbb{K}PS^1$ is the hyperbolic space on $\mathbb{K}$. (In the terminology of Gilmore [59] we have more generally $\mathbb{K}PS^n = \mathrm{U}(n, 1; \mathbb{K})/\mathrm{U}(n; \mathbb{K}) \times \mathrm{U}(1; \mathbb{K})$.)

Let us now restrict our attention to the fibrations $f: M \underset{F}{\rightarrow} N$ for the cases $2m = 4$ and 8. In these cases, F is a Lie group and we call X_j the generator(s) of F with $j = 1$ if $2m = 4$ and $j = 1$, 2 and 3 if $2m = 8$. We thus have a second important result. If Φ is a solution of the sigma model defined on N and with values on a Riemannian symmetric space G/H, then $\Psi = \phi \circ f$ is a solution of the sigma model defined on M and with values on G/H. Furthermore, the solution Ψ is invariant under F so that $X_j \Psi = 0$ for each j. Our second result can be shown as

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "\Psi = \phi \circ f"] & \\ N \arrow[r, "\phi"'] & \mathrm{G/H}. \end{tikzcd}$$

Finally, it is to be mentioned that this second result leads to non-trivial analogues of the passage formulae introduced in [17] and worked out further in [60].

7.3. Towards future investigations

7.3.1. PseudoHurwitz transformations. In the last analysis, the quasiHurwitz and Hurwitz transformations may be regarded as special cases of transformations of the type $\boldsymbol{x} = A(\boldsymbol{u})\varepsilon\boldsymbol{u}$. They follow from specific choices for the matrix $\varepsilon = \mathrm{diag}(1, \varepsilon_1, \ldots, \varepsilon_{2m-1})$ with $\varepsilon_\alpha = +1, 1 \leqslant \alpha \leqslant 2m-1$. The transformations which are neither Hurwitz nor quasiHurwitz transformations are called pseudoHurwitz transformations [61]. For $2m = 2$ and 4, the latter transformations do not lead to something new but, for $2m = 8$, we have a preliminary result. In this case, the pseudoHurwitz transformations correspond to $\mathbb{R}^8 \rightarrow \mathbb{R}^7$ maps which provide *explicit* realisations for the Hopf fibration $S^7 \rightarrow \mathbb{C}P^3$ of compact fibre S^1 and its non-compact analogues, namely (up to homeomorphisms), $\mathbb{R}^4 \times S^3 \rightarrow \mathbb{R}^4 \times S^2$ of compact fibre S^1 and $\mathbb{R}^4 \times S^3 \rightarrow \mathbb{R}^3 \times S^3$ of non-compact fibre $\mathbb{R}$. We note that the last two fibrations are new and that the fibration $S^7 \rightarrow \mathbb{C}P^3$ is of central importance in twistor theory.

7.3.2. Canonical transformations. The quasiHurwitz and Hurwitz maps correspond to transformations of a 'distance' in $\mathbb{R}^{2m-n}$ into (the square of) a 'distance' in $\mathbb{R}^{2m}$. A question now arises. What is the parentage of the Hurwitz (and quasiHurwitz) transformations with canonical transformations? A partial answer appears in [31], especially for the KS transformation. We give here a property which might be a good starting point for studying the link between Hurwitz transformations, canonical transformations and homogeneous canonical transformations developed by Dirac (see [62]). For fixed $(2m, 2m-n)$, let us define the $2m-n$ components $\mathrm{d}q_i$ by $\mathrm{d}x_i$ and the $2m$ components $\mathrm{d}Q_\alpha$ by $\mathrm{d}u_\alpha$. Similarly, let $p_i = \partial/\partial x_i$ and $P_\alpha = \partial/\partial u_\alpha$. Then we can derive the property that

$$\sum_i \mathrm{d}q_i\, p_i = \sum_\alpha \mathrm{d}Q_\alpha\, P_\alpha$$

where the sum over i extends on $2m-n$ values and the one over α on $2m$ values with $(2m, 2m-n) = (8, 5)$, $(4, 3)$ and $(2, 1)$. We note that the latter relation corresponds to a (bijective) canonical transformation in classical mechanics. We also note that the n constraint conditions $X_j = 0$ correspond to primary first-class constraints in the sense of Dirac [62].

7.3.3. Path integral methods. In recent years, path integral techniques have been applied, in conjunction with the use of the compact Hurwitz transformation for $2m = 4$, to various potentials (see, for instance, [18-23, 57]). When using such techniques, care must be exercised in the treatment of the time variable (see the paper [18] by Young and DeWitt-Morette). With a specific definition of the time variable, it should be also possible to use [63], in the framework of path integral methods, the other Hurwitz transformations introduced in this work.

7.3.4. Hyperspherical harmonics. It has been shown in [17] how the KS transformation allows us to construct $\mathbb{R}^4$ hyperspherical harmonics from $\mathbb{R}^3$ spherical harmonics. Indeed, such a construction is a consequence of the connection between Laplace operators associated to the chain $\mathrm{O}(3) \subset \mathrm{O}(4)$ inherent to the compact Hurwitz transformation for $2m = 4$. Along the same line, the other Hurwitz transformations yield connections between Laplace and/or d'Alembert operators which could be used to find relations between certain generalised hyperbolic harmonics. These matters are currently under study in relation to the Schwinger and Bargmann generating function methods [64].

7.3.5. Other pseudoHopf fibrations. Finally, we would like to mention, as a possible pending part of this work, the study of two pseudoHopf fibrations on hyperboloids (one with a compact fibre and another one with a non-compact fibre) that parallelise the Hopf fibration on spheres $S^{15} \to S^8$ of compact fibre S^7.

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the aid of the algebraic and symbolic programming system REDUCE. Information concerning the coding with REDUCE of most of the algebraic facets of this paper can be obtained from MK.

Appendix 1. Clifford matrices for the Cayley–Dickson algebras $A(c_1, c_2, c_3)$

It is a simple matter of matrix calculation to obtain from (40) the seven matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_7$ associated with the eight-dimensional Cayley-Dickson algebra $A(c_1, c_2, c_3)$. These Clifford matrices can be written as:

$$\Gamma_1 = E_{12} + c_1 E_{21} + E_{34} + c_1 E_{43} + E_{56} + c_1 E_{65} - E_{78} - c_1 E_{87}$$

$$\Gamma_2 = E_{13} + c_2 E_{31} - E_{24} - c_2 E_{42} + E_{57} + c_2 E_{75} + E_{68} + c_2 E_{86}$$

$$\Gamma_3 = E_{14} - c_1 c_2 E_{41} - c_1 E_{23} + c_2 E_{32} + E_{58} - c_1 c_2 E_{85} + c_1 E_{67} - c_2 E_{76}$$

$$\Gamma_4 = E_{15} + c_3 E_{51} - E_{26} - c_3 E_{62} - E_{37} - c_3 E_{73} - E_{48} - c_3 E_{84}$$

$$\Gamma_5 = E_{16} - c_1 c_3 E_{61} - c_1 E_{25} + c_3 E_{52} - E_{38} + c_1 c_3 E_{83} - c_1 E_{47} + c_3 E_{74}$$

$$\Gamma_6 = E_{17} - c_2 c_3 E_{71} + E_{28} - c_2 c_3 E_{82} - c_2 E_{35} + c_3 E_{53} + c_2 E_{46} - c_3 E_{64}$$

$$\Gamma_7 = E_{18} + c_1 c_2 c_3 E_{81} + c_1 E_{27} + c_2 c_3 E_{72} - c_2 E_{36} - c_1 c_3 E_{63} + c_1 c_2 E_{45} + c_3 E_{54}$$

where E_{ab} stands for the matrix with the elements $(E_{ab})_{\alpha\beta} = \delta(a, \alpha)\delta(b, \beta)$. Similarly, the three Clifford matrices Γ_1, Γ_2 and Γ_3 for the four-dimensional algebra $A(c_1, c_2)$ are

$$\Gamma_1 = E_{12} + c_1 E_{21} + E_{34} + c_1 E_{43}$$

$$\Gamma_2 = E_{13} + c_2 E_{31} - E_{24} - c_2 E_{42}$$

$$\Gamma_3 = E_{14} - c_1 c_2 E_{41} - c_1 E_{23} + c_2 E_{32}$$

and the (sole) Clifford matrix Γ_1 for the two-dimensional algebra $A(c_1)$ is

$$\Gamma_1 = E_{12} + c_1 E_{21}.$$

We now consider the three cases $2m = 2$, 4 and 8 in a global way. For fixed $2m$ ($=2$, 4 or 8), the matrices Γ_k of order $2m$ span a Clifford algebra of degree $2m-1$. Such an algebra is either $\mathscr{C}(0, 2m-1)$ or $\mathscr{C}(m, m-1)$. Following Deepak *et al* [50] we may associate to each of the latter two Clifford algebras a Dirac group of order 2^{2m}. For fixed $2m$, the corresponding Dirac groups have $2^{2m-1}+2$ conjugation classes. In addition, these groups have 2^{2m-1} irreducible representations of dimension 1; and two irreducible representations of dimension 2^{m-1}.

After completion of this work, we were made aware of a paper by Deming Li *et al* [51] on a general method of generating and classifying Clifford algebras.

Appendix 2. Inverses of the Hurwitz transformations

In order to further understand the fibrations described in § 5 and to facilitate the use of the Hurwitz transformations in physical applications, we list (without proof) in this appendix the inverses of typical Hurwitz transformations in the two-, four- and eight-dimensional cases. The method employed for deriving these inverse transformations is an adaptation of the one developed by Kustaanheimo and Stiefel [7] in the four-dimensional compact case (see also [27]).

Case 2m = 8. Let us consider the right Hurwitz transformation $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$, see (61). We want to find from (61) the reciprocal image $(u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ in $\mathbb{R}^8$ of $(x_0, x_4, x_5, x_6, x_7)$ in $\mathbb{R}^5$. Equation (61) leads to

$$\begin{aligned} u_4 &= (x_4u_0 - c_1x_5u_1 - c_2x_6u_2 + c_1c_2x_7u_3)/(2\rho) \\ u_5 &= (x_5u_0 - x_4u_1 - c_2x_7u_2 + c_2x_6u_3)/(2\rho) \\ u_6 &= (x_6u_0 + c_1x_7u_1 - x_4u_2 - c_1x_5u_3)/(2\rho) \\ u_7 &= (x_7u_0 + x_6u_1 - x_5u_2 - x_4u_3)/(2\rho) \end{aligned} \tag{A2.1}$$

for $\rho = u_0^2 - c_1u_1^2 - c_2u_2^2 + c_1c_2u_3^2 \neq 0$. The quantity ρ is formally given in terms of the data x_0, x_4, x_5, x_6 and x_7 by $2\rho = x_0 \pm N(x)$, where we assume that $(x_0, x_4, x_5, x_6, x_7)$ satisfies the condition

$$N(x)^2 = x_0^2 - c_3x_4^2 + c_1c_3x_5^2 + c_2c_3x_6^2 - c_1c_2c_3x_7^2 > 0.$$

In the compact case $c_1 = c_2 = c_3 = -1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$ may be obtained from (A2.1) with

$$c_1 = -1 \qquad c_2 = -1 \qquad 2\rho = x_0 + (x_0^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2)^{1/2}$$
$$u_0 = \sqrt{\rho}\cos\phi\cos\chi\cos\Psi \qquad u_1 = \sqrt{\rho}\cos\phi\cos\chi\sin\Psi$$
$$u_2 = \sqrt{\rho}\cos\phi\sin\chi \qquad u_3 = \sqrt{\rho}\sin\phi$$

where ϕ, χ and Ψ are real parameters exhibiting an S^3 fibre.

In the non-compact case $c_1 = c_2 = -c_3 = -1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$ may be obtained from (A2.1) with

$$c_1 = -1 \qquad c_2 = -1 \qquad 2\rho = x_0 \pm (x_0^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2)^{1/2}$$
$$x_0 > 0 \qquad u_0 = \sqrt{\rho}\cos\phi\cos\chi\cos\Psi \qquad u_1 = \sqrt{\rho}\cos\phi\cos\chi\sin\Psi$$
$$u_2 = \sqrt{\rho}\cos\phi\sin\chi \qquad u_3 = \sqrt{\rho}\sin\phi$$

where ϕ, χ and Ψ are real parameters exhibiting an S^3 fibre.

In the non-compact case $c_1 = c_2 = c_3 = 1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$ may be obtained from (A2.1) with

$$c_1 = 1 \qquad c_2 = 1 \qquad 2\rho = x_0 \pm (x_0^2 - x_4^2 + x_5^2 + x_6^2 - x_7^2)^{1/2}$$

and

$$\left.\begin{aligned} u_0 &= \sqrt{\rho}\cos\phi\cosh\chi \qquad & u_1 &= \sqrt{\rho}\cos\Psi\sinh\chi \\ u_2 &= \sqrt{\rho}\sin\Psi\sinh\chi \qquad & u_3 &= \sqrt{\rho}\sin\phi\cosh\chi \end{aligned}\right\} \quad \text{if } \rho > 0$$

or

$$\left.\begin{aligned} u_0 &= \sqrt{-\rho}\cos\Psi\sinh\chi \qquad & u_1 &= \sqrt{-\rho}\cos\phi\cosh\chi \\ u_2 &= \sqrt{-\rho}\sin\phi\cosh\chi \qquad & u_3 &= \sqrt{-\rho}\sin\Psi\sinh\chi \end{aligned}\right\} \quad \text{if } \rho < 0$$

where ϕ, χ and Ψ are real parameters exhibiting an $\mathbb{R}^2 \times S^1$ fibre. The five remaining choices for c_i ($i = 1, 2$ and 3) yield results similar to the ones for the non-compact case $c_1 = c_2 = c_3 = 1$.

Case 2m = 4. We choose the right Hurwitz transformation $x = \mathscr{H}_{\mathrm{R}}^{(3)}(u)$, see (64). Let us look, from (64), for the reciprocal image (u_0, u_1, u_2, u_3) in $\mathbb{R}^4$ of (x_0, x_1, x_2) in $\mathbb{R}^3$. Equation (64) leads to

$$u_1 = (x_1 u_0 - c_2 x_2 u_3)/(2\rho) \qquad u_2 = (x_2 u_0 + c_1 x_1 u_3)/(2\rho) \tag{A2.2}$$

for $\rho = u_0^2 + c_1 c_2 u_3^2 \neq 0$. The quantity ρ is formally given in terms of the data x_0, x_1 and x_2 by $2\rho = x_0 \pm N(x)$, where we assume that (x_0, x_1, x_2) satisfies the condition

$$N(x)^2 = x_0^2 - c_1 x_1^2 - c_2 x_2^2 > 0.$$

In the compact case $c_1 = c_2 = -1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(3)}(u)$ may be obtained from (A2.2) with

$$c_1 = -1 \qquad c_2 = -1 \qquad 2\rho = x_0 + (x_0^2 + x_1^2 + x_2^2)^{1/2}$$
$$u_0 = \sqrt{\rho} \cos \phi \qquad u_3 = \sqrt{\rho} \sin \phi$$

where ϕ is a real parameter exhibiting an S^1 fibre.

In the non-compact case $c_1 = c_2 = 1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(3)}(u)$ may be obtained from (A2.2) with

$$c_1 = 1 \qquad c_2 = 1 \qquad 2\rho = x_0 \pm (x_0^2 - x_1^2 - x_2^2)^{1/2}$$
$$x_0 > 0 \qquad u_0 = \sqrt{\rho} \cos \phi \qquad u_3 = \sqrt{\rho} \sin \phi$$

where ϕ is a real parameter exhibiting an S^1 fibre.

In the non-compact case $c_1 = -c_2 = -1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(3)}(u)$ may be obtained from (A2.2) with

$$c_1 = -1 \qquad c_2 = 1 \qquad 2\rho = x_0 \pm (x_0^2 + x_1^2 - x_2^2)^{1/2}$$

and

$$u_0 = \sqrt{\rho} \cosh \phi \qquad u_3 = \sqrt{\rho} \sinh \phi \qquad \text{if } \rho > 0$$

or

$$u_0 = \sqrt{-\rho} \sinh \phi \qquad u_3 = \sqrt{-\rho} \cosh \phi \qquad \text{if } \rho < 0$$

where ϕ is a real parameter exhibiting an $\mathbb{R}$ fibre. The non-compact case $c_1 = -c_2 = 1$ is very similar to the case $c_1 = -c_2 = -1$.

Case 2m = 2. The sole possibility is $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u) = \mathscr{H}_{\mathrm{L}}^{(1)}(u) = \mathscr{H}^{(0)}(u)$, see (68). The reciprocal image (u_0, u_1) in $\mathbb{R}^2$ of x_0 in $\mathbb{R}^+$ or $\mathbb{R}$ is readily obtained from (68). In the compact case $c_1 = -1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$ is given by

$$u_0 = \sqrt{x_0} \cos \phi \qquad u_1 = \sqrt{x_0} \sin \phi \qquad \text{for } x_0 \geqslant 0$$

where ϕ is a real parameter exhibiting an S^1 fibre. In the non-compact case $c_1 = 1$, the reverse of $x = \mathscr{H}_{\mathrm{R}}^{(1)}(u)$ is given by

$$u_0 = \sqrt{x_0} \cosh \phi \qquad u_1 = \sqrt{x_0} \sinh \phi \qquad \text{for } x_0 > 0$$

or

$$u_0 = \sqrt{-x_0}\sinh\phi \qquad u_1 = \sqrt{-x_0}\cosh\phi \qquad \text{for } x_0 < 0$$

where ϕ is a real parameter exhibiting an $\mathbb{R}$ fibre.

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